Essay on the Unified Theory of the Classical Fields of Gravitation and Electromagnetism

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version 1.0, March 2002

Extract

We would like to introduce a new attempt to establish a unified theory of the classical fields of gravitation and electromagnetism which complies with the geometric paradigm of the theory of relativity of Albert Einstein. The theory we shall describe is geometrically unified in the sense that the fields equations, and the Hamiltonian function from which they derive, are formally unified entities (i.e. they are not the sum of several independent parts) which depend only on the metric of the particular spacetime being considered.

Remarks

In this essay, we make use of well known results from the theory of relativity of Albert Einstein without giving explicit references. We implicitly refer to [1] for a detailed explanation of these results.

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1 Introduction

The attempts to geometrically unify the classical fields of gravitation and electromagnetism can be classified in two categories (see [1]):

- those consisting in abandoning the Riemannian geometry in the 4-dimensional spacetime by considering geometries with affine connections (symmetric or asymmetric). This is the case with the attempts of H. Weyl, A.S. Eddington, E. Schrödinger and A. Einstein; and
- those consisting in keeping the Riemannian geometry by increasing the dimension of the spacetime being considered. This is the case with the attempts of T. Kaluza, O. B. Klein and A. Einstein.

Our approach belongs to the second category by considering a "double" 4-dimensional spacetime composed of the usual 4-dimensional spacetime and the corresponding dual 4-dimensional spacetime.

The linearization of the fields equations by the so-called "Einstein-Infeld-Hoffmann" (EIH) method, will allow us to:

- deduct the "Newton-Maxwell-Lorentz" results of classical physics. That is, the gravitational field and the electromagnetic field of charged mass points and their equations of motion;
- explain why the Lorentz transformation must be linear in classical physics. This property is imposed in classical physics by the sake of simplicity, without being formally justified as it is the case in the theory we shall describe; and
- show that the electromagnetic field could not exist without the gravitational field and by consequence, that we are not allowed to describe the electromagnetic field without considering the gravitational field.

It is important to say in advance that these deductions will hold separately in each of the two 4-dimensional spacetime being considered. During the linearization process, we will see that by imposing a formal restriction to the equations and to their solutions, the things happen as if there was only one 4-dimensional spacetime to consider.

2 The structure of the spacetime and the metric

The structure of the spacetime on which the theory is based is double, for it is composed of the usual 4-dimensional spacetime (z^1, z^2, z^3, z^4) and the corresponding dual 4-dimensional spacetime $\left(z^{\bar 1}, z^{\bar 2}, z^{\bar 3}, z^{\bar 4}\right)$.

z^1, z^2, z^3 represent the usual spacial coordinates x^1, x^2, x^3 and $z^{\bar 1}, z^{\bar 2}, z^{\bar 3}$ represent the corresponding dual spacial coordinates $x^{\bar 1}, x^{\bar 2}, x^{\bar 3}$. In order to facilitate the calculations, z^4 which represents the usual time coordinate will be chosen imaginary: $z^4 = ict$. The same for $z^{\bar 4}$ which represents the corresponding dual time coordinate: $z^{\bar 4} = ic\bar{t}$.

This particular structure of the spacetime allows the introduction of the following metric:

$$ds^2 = dz_A dz^A = g_{AB} dz^A dz^B = g_{\alpha\beta} dz^\alpha dz^\beta + g_{\alpha\bar\beta} dz^\alpha dz^{\bar\beta} + g_{\bar\alpha\beta} dz^{\bar\alpha} dz^\beta + g_{\bar\alpha\bar\beta} dz^{\bar\alpha} dz^{\bar\beta} \tag{1}$$

with $g_{AB} = g_{BA}$ symmetric , $g_{AB} g^{BC} = \delta_A^C$, $dz_A = g_{AB} dz^B$ and $z^A = (z^\alpha, z^{\bar\alpha})$

Remark:

The following notations are used in this essay:

- Lower Latin indices run over the values from 1 to 3 (spatial coordinates), Greek indices run over the values from 1 to 4 (spatial and time coordinates) and upper Latin indices run over the values from the usual and the corresponding dual Greek indices. Repeated dummy indices indicate an implicit summation:
  - $g_{ab} dz^a dz^b = \sum_{a=1}^{3} \sum_{b=1}^{3} g_{ab} dz^a dz^b$, $g_{a\bar b} dz^a dz^{\bar b} = \sum_{a=1}^{3} \sum_{\bar b=\bar 1}^{\bar 3} g_{a\bar b} dz^a dz^{\bar b}$
  - $g_{\alpha\beta} dz^\alpha dz^\beta = \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} g_{\alpha\beta} dz^\alpha dz^\beta$, $g_{\alpha\bar\beta} dz^\alpha dz^{\bar\beta} = \sum_{\alpha=1}^{4} \sum_{\bar\beta=\bar 1}^{\bar 4} g_{\alpha\bar\beta} dz^\alpha dz^{\bar\beta}$
  - $g_{AB} dz^A dz^B = g_{\alpha\beta} dz^\alpha dz^\beta + g_{\alpha\bar\beta} dz^\alpha dz^{\bar\beta} + g_{\bar\alpha\beta} dz^{\bar\alpha} dz^\beta + g_{\bar\alpha\bar\beta} dz^{\bar\alpha} dz^{\bar\beta}$
- $\partial_A f = \dfrac{\partial f}{\partial z^A}$, $\partial_\alpha f = \dfrac{\partial f}{\partial z^\alpha}$, $\partial_{\bar\alpha} f = \dfrac{\partial f}{\partial z^{\bar\alpha}}$ are partial derivatives
- $D_A f$, $D_\alpha f$, $D_{\bar\alpha} f$ are covariant derivatives
- $z^{\bar\alpha}$ is the corresponding dual vector of the usual vector z^α
- $z^A = (z^\alpha, z^{\bar\alpha})$ with $z^\alpha = (z^1, z^2, z^3, z^4)$ and $z^{\bar\alpha} = \left(z^{\bar 1}, z^{\bar 2}, z^{\bar 3}, z^{\bar 4}\right)$
- $\acute{z}^A = \acute{z}^A\left(z^B\right)$ with $\acute{z}^\alpha = \acute{z}^\alpha\left(z^\beta\right)$ and $\acute{z}^{\bar\alpha} = \acute{z}^{\bar\alpha}\left(z^{\bar\beta}\right)$ are coordinate transformations

3 The coordinate transformations and the tensors

The coordinate transformations that we shall consider are of the form $\acute{z}^A = \acute{z}^A\left(z^B\right)$ with $\acute{z}^\alpha = \acute{z}^\alpha(z^\beta)$ and $\acute{z}^{\bar\alpha} = \acute{z}^{\bar\alpha}(z^{\bar\beta})$.

Under these particular coordinate transformations, a contravariant vector V^A will transform according to:

$$\acute{V}^A = V^K \frac{d\acute{z}^A}{dz^K} \Rightarrow \acute{V}^\alpha = V^K \frac{d\acute{z}^\alpha}{dz^K} = V^\epsilon \frac{d\acute{z}^\alpha}{dz^\epsilon} + V^{\bar\epsilon}\overbrace{\frac{d\acute{z}^\alpha}{dz^{\bar\epsilon}}}^{=0} \Rightarrow \acute{V}^\alpha = V^\epsilon \frac{d\acute{z}^\alpha}{dz^\epsilon} \Rightarrow \acute{V}^{\bar\alpha} = V^{\bar\epsilon}\frac{d\acute{z}^{\bar\alpha}}{dz^{\bar\epsilon}}$$

A covariant vector V_A will transform according to:

$$\acute{V}_A = V_K \frac{dz^K}{d\acute{z}^A} \Rightarrow \acute{V}_\alpha = V_K \frac{dz^K}{d\acute{z}^\alpha} = V_\epsilon \frac{dz^\epsilon}{d\acute{z}^\alpha} + V_{\bar\epsilon}\overbrace{\frac{dz^{\bar\epsilon}}{d\acute{z}^\alpha}}^{=0} \Rightarrow \acute{V}_\alpha = V_\epsilon \frac{dz^\epsilon}{d\acute{z}^\alpha} \Rightarrow \acute{V}_{\bar\alpha} = V_{\bar\epsilon}\frac{dz^{\bar\epsilon}}{d\acute{z}^{\bar\alpha}}$$

The generalization to tensors of any rank is immediate:

$$\acute{T}^{AB.D}_{..C} = T^{KL.N}_{..M}\frac{d\acute{z}^A}{dz^K}\frac{d\acute{z}^B}{dz^L}\frac{dz^M}{d\acute{z}^C}\frac{d\acute{z}^D}{dz^N} \quad , \quad \acute{T}^{\alpha\bar\beta.\delta}_{..\gamma} = T^{\epsilon\bar\theta.\sigma}_{..\tau}\frac{d\acute{z}^\alpha}{dz^\epsilon}\frac{d\acute{z}^{\bar\beta}}{dz^{\bar\theta}}\frac{dz^\tau}{d\acute{z}^\gamma}\frac{d\acute{z}^\delta}{dz^\sigma}$$

We may also mix upper Latin indices and Greek indices:

$$\acute{T}^{A\bar\alpha.\beta}_{..B} = T^{K\bar\epsilon.\theta}_{..L}\frac{d\acute{z}^A}{dz^K}\frac{d\acute{z}^{\bar\alpha}}{dz^{\bar\epsilon}}\frac{dz^L}{d\acute{z}^B}\frac{d\acute{z}^\beta}{dz^\theta}$$

The inverse transformation is similar:

$$T^{A\bar\alpha.\beta}_{..B} = \acute{T}^{K\bar\epsilon.\theta}_{..L}\frac{dz^A}{d\acute{z}^K}\frac{dz^{\bar\alpha}}{d\acute{z}^{\bar\epsilon}}\frac{d\acute{z}^L}{dz^B}\frac{dz^\beta}{d\acute{z}^\theta}$$

The particular case of the metric tensor:

When we consider the metric tensor g_{AB}, we obtain:

$$\acute{g}_{AB} = g_{KL}\frac{dz^K}{d\acute{z}^A}\frac{dz^L}{d\acute{z}^B} \Rightarrow \acute{g}_{\alpha\beta} = g_{\epsilon\theta}\frac{dz^\epsilon}{d\acute{z}^\alpha}\frac{dz^\theta}{d\acute{z}^\beta} \Rightarrow \acute{g}_{\alpha\bar\beta} = g_{\epsilon\bar\theta}\frac{dz^\epsilon}{d\acute{z}^\alpha}\frac{dz^{\bar\theta}}{d\acute{z}^{\bar\beta}} \qquad (2)$$

These relations mean that $\acute{g}_{\alpha\beta}$ depend only on $g_{\epsilon\theta}$ and that $\acute{g}_{\alpha\bar\beta}$ depend only on $g_{\epsilon\bar\theta}$. In particular, if $g_{\epsilon\bar\theta} = 0$, we obtain $\acute{g}_{\alpha\bar\beta} = 0$ whatever $g_{\epsilon\theta}$. It is important to notice that these results are due to the particular coordinate transformations being considered.

4 The Christoffel symbols

The calculation $\int \delta ds = 0$ (the variation is operated on the z^A), with $ds^2 = g_{AB} dz^A dz^B$ leads to the following equation:

$$\frac{d^2 z^C}{d^2 s} + \Gamma^C_{AB} \frac{dz^A}{ds} \frac{dz^B}{ds} = 0 \ , \ \Gamma^C_{AB} = \tfrac{1}{2} g^{CK} \left(\partial_A g_{BK} + \partial_B g_{AK} - \partial_K g_{AB} \right) \ , \ \Gamma^C_{AB} = \Gamma^C_{BA} \text{ symmetric} \quad (3)$$

The Christoffel symbols Γ^C_{AB} can be developed into Greek indices:

$$\Gamma^{\gamma}_{\alpha\beta} = \tfrac{1}{2} g^{\gamma K} \left(\partial_\alpha g_{\beta K} + \partial_\beta g_{\alpha K} - \partial_K g_{\alpha\beta} \right)$$
$$\Rightarrow \Gamma^{\gamma}_{\alpha\beta} = \tfrac{1}{2} g^{\gamma\epsilon} \left(\partial_\alpha g_{\beta\epsilon} + \partial_\beta g_{\alpha\epsilon} - \partial_\epsilon g_{\alpha\beta} \right) + \tfrac{1}{2} g^{\gamma\bar\epsilon} \left(\partial_\alpha g_{\beta\bar\epsilon} + \partial_\beta g_{\alpha\bar\epsilon} - \partial_{\bar\epsilon} g_{\alpha\beta} \right)$$

$$\Gamma^{\gamma}_{\alpha\bar\beta} = \tfrac{1}{2} g^{\gamma K} \left(\partial_\alpha g_{\bar\beta K} + \partial_{\bar\beta} g_{\alpha K} - \partial_K g_{\alpha\bar\beta} \right)$$
$$\Rightarrow \Gamma^{\gamma}_{\alpha\bar\beta} = \tfrac{1}{2} g^{\gamma\epsilon} \left(\partial_\alpha g_{\bar\beta\epsilon} + \partial_{\bar\beta} g_{\alpha\epsilon} - \partial_\epsilon g_{\alpha\bar\beta} \right) + \tfrac{1}{2} g^{\gamma\bar\epsilon} \left(\partial_\alpha g_{\bar\beta\bar\epsilon} + \partial_{\bar\beta} g_{\alpha\bar\epsilon} - \partial_{\bar\epsilon} g_{\alpha\bar\beta} \right)$$

$$\Gamma^{\gamma}_{\bar\alpha\beta} = \tfrac{1}{2} g^{\gamma K} \left(\partial_{\bar\alpha} g_{\beta K} + \partial_\beta g_{\bar\alpha K} - \partial_K g_{\bar\alpha\beta} \right)$$
$$\Rightarrow \Gamma^{\gamma}_{\bar\alpha\beta} = \tfrac{1}{2} g^{\gamma\epsilon} \left(\partial_{\bar\alpha} g_{\beta\epsilon} + \partial_\beta g_{\bar\alpha\epsilon} - \partial_\epsilon g_{\bar\alpha\beta} \right) + \tfrac{1}{2} g^{\gamma\bar\epsilon} \left(\partial_{\bar\alpha} g_{\beta\bar\epsilon} + \partial_\beta g_{\bar\alpha\bar\epsilon} - \partial_{\bar\epsilon} g_{\bar\alpha\beta} \right)$$

$$\Gamma^{\gamma}_{\bar\alpha\bar\beta} = \tfrac{1}{2} g^{\gamma K} \left(\partial_{\bar\alpha} g_{\bar\beta K} + \partial_{\bar\beta} g_{\bar\alpha K} - \partial_K g_{\bar\alpha\bar\beta} \right)$$
$$\Rightarrow \Gamma^{\gamma}_{\bar\alpha\bar\beta} = \tfrac{1}{2} g^{\gamma\epsilon} \left(\partial_{\bar\alpha} g_{\bar\beta\epsilon} + \partial_{\bar\beta} g_{\bar\alpha\epsilon} - \partial_\epsilon g_{\bar\alpha\bar\beta} \right) + \tfrac{1}{2} g^{\gamma\bar\epsilon} \left(\partial_{\bar\alpha} g_{\bar\beta\bar\epsilon} + \partial_{\bar\beta} g_{\bar\alpha\bar\epsilon} - \partial_{\bar\epsilon} g_{\bar\alpha\bar\beta} \right)$$

And so forth for $\Gamma^{\bar\gamma}_{\bar\alpha\bar\beta}$, $\Gamma^{\bar\gamma}_{\bar\alpha\beta}$, $\Gamma^{\bar\gamma}_{\alpha\bar\beta}$, $\Gamma^{\bar\gamma}_{\alpha\beta}$.

5 The covariant derivatives

The covariant derivatives are expressed by:

$$\begin{array}{lcl}
D_C V^A = \partial_C V^A + \Gamma^A_{KC} V^K & , & D_C V_A = \partial_C V_A - \Gamma^K_{AC} V_K \\
D_C T^{AB} = \partial_C T^{AB} + \Gamma^A_{KC} T^{KB} + \Gamma^B_{KC} T^{AK} & , & D_C T_{AB} = \partial_C T_{AB} - \Gamma^K_{AC} T_{KB} - \Gamma^K_{BC} T_{AK} \\
D_C T^B_A = \partial_C A^B_A - \Gamma^K_{AC} T^B_K + \Gamma^B_{KC} T^K_A & , & D_C \left(V^A V^B \right) = D_C V^A V^B + V^A D_C V^B \\
D_C \left(V^A V_B \right) = D_C V^A V_B + V^A D_C V_B & , & D_C \left(V_A V_B \right) = D_C V_A V_B + V_A D_C V_B
\end{array}$$

These expressions can be developed into Greek indices:

$$D_\gamma T^{\alpha\beta} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha_{K\gamma} T^{K\beta} + \Gamma^\beta_{K\gamma} T^{\alpha K} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha_{\epsilon\gamma} T^{\epsilon\beta} + \Gamma^\alpha_{\bar\epsilon\gamma} T^{\bar\epsilon\beta} + \Gamma^\beta_{\epsilon\gamma} T^{\alpha\epsilon} + \Gamma^\beta_{\bar\epsilon\gamma} T^{\alpha\bar\epsilon}$$
$$D_{\bar\gamma} T^{\alpha\beta} = \partial_{\bar\gamma} T^{\alpha\beta} + \Gamma^\alpha_{K\bar\gamma} T^{K\beta} + \Gamma^\beta_{K\bar\gamma} T^{\alpha K} = \partial_{\bar\gamma} T^{\alpha\beta} + \Gamma^\alpha_{\epsilon\bar\gamma} T^{\epsilon\beta} + \Gamma^\alpha_{\bar\epsilon\bar\gamma} T^{\bar\epsilon\beta} + \Gamma^\beta_{\epsilon\bar\gamma} T^{\alpha\epsilon} + \Gamma^\beta_{\bar\epsilon\bar\gamma} T^{\alpha\bar\epsilon}$$
$$D_\gamma \left(V^\alpha V^\beta \right) = D_\gamma V^\alpha V^\beta + V^\alpha D_\gamma V^\beta$$

6 The covariant derivatives of the metric tensor

When the Christoffel symbols are expressed by (3), we have:

$$D_C g_{AB} = D_C g^{AB} = 0 \tag{4}$$

This relation can be developed into Greek indices:

$$D_\gamma g_{\alpha\beta} = D_{\bar\gamma} g_{\alpha\beta} = D_\gamma g_{\alpha\bar\beta} = D_{\bar\gamma} g_{\alpha\bar\beta} = D_\gamma g^{\alpha\beta} = D_{\bar\gamma} g^{\alpha\beta} = D_\gamma g^{\alpha\bar\beta} = D_{\bar\gamma} g^{\alpha\bar\beta} = 0$$

7 The Riemann and the Ricci curvature tensors

The calculation $T_{CAB} = D_C D_A V_B - D_A D_C V_B$ leads to the result $T_{CAB} = R^D_{CAB} V_D$
where $R^D_{CAB} = \partial_A \Gamma^D_{BC} - \partial_C \Gamma^D_{AB} + \Gamma^D_{AL}\Gamma^L_{CB} - \Gamma^L_{AB}\Gamma^D_{LC}$ is the Riemann curvature tensor.

By contracting the Riemann curvature tensor, we obtain the Ricci curvature tensor:

$$R_{AB} = R^K_{KAB} = \partial_A \Gamma^K_{BK} - \partial_K \Gamma^K_{AB} + \Gamma^K_{AL}\Gamma^L_{KB} - \Gamma^L_{AB}\Gamma^K_{LK} \text{ , symmetric} \tag{5}$$

The symmetry of R_{AB} follows from the fact that $\Gamma^K_{AK} = \frac{1}{2}\partial_A \ln|g|$, where g = determinant of $g_{AB} = |g_{AB}|$

R_{AB} can be developed into Greek indices:

$$\begin{aligned}
R_{\alpha\beta} &= \partial_\alpha \Gamma^K_{\beta K} - \partial_K \Gamma^K_{\alpha\beta} + \Gamma^K_{\alpha L}\Gamma^L_{K\beta} - \Gamma^L_{\alpha\beta}\Gamma^K_{LK} \\
&\Rightarrow R_{\alpha\beta} = \partial_\alpha \Gamma^\epsilon_{\beta\epsilon} + \partial_\alpha \Gamma^{\bar\epsilon}_{\beta\bar\epsilon} - \partial_\epsilon \Gamma^\epsilon_{\alpha\beta} - \partial_{\bar\epsilon} \Gamma^{\bar\epsilon}_{\alpha\beta} + \Gamma^\epsilon_{\alpha L}\Gamma^L_{\epsilon\beta} + \Gamma^{\bar\epsilon}_{\alpha L}\Gamma^L_{\bar\epsilon\beta} - \Gamma^L_{\alpha\beta}\Gamma^\epsilon_{L\epsilon} - \Gamma^L_{\alpha\beta}\Gamma^{\bar\epsilon}_{L\bar\epsilon} \\
&\Rightarrow R_{\alpha\beta} = \partial_\alpha \Gamma^\epsilon_{\beta\epsilon} + \partial_\alpha \Gamma^{\bar\epsilon}_{\beta\bar\epsilon} - \partial_\epsilon \Gamma^\epsilon_{\alpha\beta} - \partial_{\bar\epsilon} \Gamma^{\bar\epsilon}_{\alpha\beta} + \Gamma^\epsilon_{\alpha\theta}\Gamma^\theta_{\epsilon\beta} + \Gamma^\epsilon_{\alpha\bar\theta}\Gamma^{\bar\theta}_{\epsilon\beta} + \Gamma^{\bar\epsilon}_{\alpha\theta}\Gamma^\theta_{\bar\epsilon\beta} + \Gamma^{\bar\epsilon}_{\alpha\bar\theta}\Gamma^{\bar\theta}_{\bar\epsilon\beta} \\
&\qquad -\Gamma^\theta_{\alpha\beta}\Gamma^\epsilon_{\theta\epsilon} - \Gamma^{\bar\theta}_{\alpha\beta}\Gamma^\epsilon_{\bar\theta\epsilon} - \Gamma^\theta_{\alpha\beta}\Gamma^{\bar\epsilon}_{\theta\bar\epsilon} - \Gamma^{\bar\theta}_{\alpha\beta}\Gamma^{\bar\epsilon}_{\bar\theta\bar\epsilon}
\end{aligned}$$

$$\begin{aligned}
R_{\alpha\bar\beta} &= \partial_\alpha \Gamma^K_{\bar\beta K} - \partial_K \Gamma^K_{\alpha\bar\beta} + \Gamma^K_{\alpha L}\Gamma^L_{K\bar\beta} - \Gamma^L_{\alpha\bar\beta}\Gamma^K_{LK} \\
&\Rightarrow R_{\alpha\bar\beta} = \partial_\alpha \Gamma^\epsilon_{\bar\beta\epsilon} + \partial_\alpha \Gamma^{\bar\epsilon}_{\bar\beta\bar\epsilon} - \partial_\epsilon \Gamma^\epsilon_{\alpha\bar\beta} - \partial_{\bar\epsilon} \Gamma^{\bar\epsilon}_{\alpha\bar\beta} + \Gamma^\epsilon_{\alpha L}\Gamma^L_{\epsilon\bar\beta} + \Gamma^{\bar\epsilon}_{\alpha L}\Gamma^L_{\bar\epsilon\bar\beta} - \Gamma^L_{\alpha\bar\beta}\Gamma^\epsilon_{L\epsilon} - \Gamma^L_{\alpha\bar\beta}\Gamma^{\bar\epsilon}_{L\bar\epsilon} \\
&\Rightarrow R_{\alpha\bar\beta} = \partial_\alpha \Gamma^\epsilon_{\bar\beta\epsilon} + \partial_\alpha \Gamma^{\bar\epsilon}_{\bar\beta\bar\epsilon} - \partial_\epsilon \Gamma^\epsilon_{\alpha\bar\beta} - \partial_{\bar\epsilon} \Gamma^{\bar\epsilon}_{\alpha\bar\beta} + \Gamma^\epsilon_{\alpha\theta}\Gamma^\theta_{\epsilon\bar\beta} + \Gamma^\epsilon_{\alpha\bar\theta}\Gamma^{\bar\theta}_{\epsilon\bar\beta} + \Gamma^{\bar\epsilon}_{\alpha\theta}\Gamma^\theta_{\bar\epsilon\bar\beta} + \Gamma^{\bar\epsilon}_{\alpha\bar\theta}\Gamma^{\bar\theta}_{\bar\epsilon\bar\beta} \\
&\qquad -\Gamma^\theta_{\alpha\bar\beta}\Gamma^\epsilon_{\theta\epsilon} - \Gamma^{\bar\theta}_{\alpha\bar\beta}\Gamma^\epsilon_{\bar\theta\epsilon} - \Gamma^\theta_{\alpha\bar\beta}\Gamma^{\bar\epsilon}_{\theta\bar\epsilon} - \Gamma^{\bar\theta}_{\alpha\bar\beta}\Gamma^{\bar\epsilon}_{\bar\theta\bar\epsilon}
\end{aligned}$$

8 The principle of equivalence

One consequence of the principle of equivalence is that we can make "appear" a gravitational field from a flat spacetime by applying a coordinate transformation. Another consequence of the principle of equivalence is that we can locally make the gravitational field "disappear" by applying an appropriate coordinate transformation.

A similar result does not exist for the electromagnetic field. We can not make appear an electromagnetic field from a flat spacetime by applying a coordinate transformation. Also, we can not locally make the electromagnetic field disappear by applying an appropriate coordinate transformation.

The metric being considered in this essay is particularly adapted to this situation. A flat spacetime is characterized by:

$$g_{AB} = \delta_{AB} \Rightarrow g_{\alpha\beta} = \delta_{\alpha\beta} \text{ and } g_{\alpha\bar{\beta}} = 0 \text{ where } \delta_{\alpha\beta} = 1 \text{ if } \alpha = \beta \text{ and } \delta_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta$$

From the equations (2), we see that when $g_{\alpha\beta} = \delta_{\alpha\beta}$, we obtain $\acute{g}_{\alpha\beta} \neq \delta_{\alpha\beta}$ when we apply a coordinate transformation. Also, we see that when $g_{\alpha\bar{\beta}} = 0$, we obtain $\acute{g}_{\alpha\bar{\beta}} = 0$ when we apply a coordinate transformation. It is therefore natural to consider $g_{\alpha\beta}$ as the gravitational field potentials and to consider $g_{\alpha\bar{\beta}}$ as the electromagnetic field potentials.

From the equations (2) and the previous paragraph, we also see that the only way to locally obtain a flat spacetime ($g_{AB} = \delta_{AB}$) by applying an appropriate coordinate transformation, is to have no electromagnetic field ($g_{\alpha\bar{\beta}} = 0$).

9 The fields equations

As the Hamiltonien function, we shall consider $H = R\sqrt{|g|}d\Omega = 0$ where $R = g^{KL}R_{KL}$,

$g =$ determinant of $g_{AB} = |g_{AB}|$ and $d\Omega = dz^1dz^2dz^3dz^4dz^{\bar{1}}dz^{\bar{2}}dz^{\bar{3}}dz^{\bar{4}}$.

The calculation $\delta H = 0$ (the variation is operated on the g_{AB}) leads to the following fields equations:

$$G_{AB} = R_{AB} - \tfrac{1}{2}g_{AB}R = 0 \text{ , with } D_K G^{AK} = 0 \text{ (from the Bianchi identities)} \tag{6}$$

For $G_{AB} = 0$, we conclude that $R = 0$ and by consequence, that $R_{AB} = 0$.

10 The linearization of the fields equations: the classical method

We shall assume that the components of the metric tensor can be expanded into a series:

$$g_{AB} = \delta_{AB} + h_{AB} = \delta_{AB} + \lambda\, h_{\underset{1}{AB}} + \lambda^2\, h_{\underset{2}{AB}} + \cdots$$

By making use of this series, the fields equations can be expanded into successive approximations. If we limit ourselves to the first approximation, we obtain:

$$\begin{aligned}
&R_{\underset{1}{AB}} = \partial_A \Gamma^L_{\underset{1}{BL}} - \partial_L \Gamma^L_{\underset{1}{AB}} \text{ with } \Gamma^L_{\underset{1}{BL}} = \tfrac{1}{2}\left(\partial_B h_{\underset{1}{LL}}\right) \text{ and } \Gamma^L_{\underset{1}{AB}} = \tfrac{1}{2}\left(\partial_A h_{\underset{1}{BL}} + \partial_B h_{\underset{1}{AL}} - \partial_L h_{\underset{1}{AB}}\right)\\
&\quad \Rightarrow R_{\underset{1}{AB}} = \tfrac{1}{2}\left(\partial_L \partial_L h_{\underset{1}{AB}} + \partial_A \partial_B h_{\underset{1}{LL}} - \partial_L \partial_A h_{\underset{1}{BL}} - \partial_L \partial_B h_{\underset{1}{AL}}\right)
\end{aligned}$$

The calculation is simplified by the introduction of the following quantities:

$$\begin{aligned}
&\gamma_{AB} = h_{AB} - \tfrac{1}{2}\delta_{AB} h \Leftrightarrow h_{AB} = \gamma_{AB} - \tfrac{1}{6}\delta_{AB}\gamma \Rightarrow h_{\underset{1}{AB}} = \gamma_{\underset{1}{AB}} - \tfrac{1}{6}\delta_{AB}\underset{1}{\gamma} \text{ (for } \delta_{KK} = 8)\\
&\quad \Rightarrow R_{\underset{1}{AB}} = \tfrac{1}{2}\left(\partial_L \partial_L \gamma_{\underset{1}{AB}} - \tfrac{1}{6}\delta_{AB}\partial_L \partial_L \underset{1}{\gamma} - \partial_L \partial_A \gamma_{\underset{1}{BL}} - \partial_L \partial_B \gamma_{\underset{1}{AL}}\right)\\
&\quad \Rightarrow \underset{1}{R} = \tfrac{1}{2}\left(-\tfrac{1}{3}\partial_L \partial_L \underset{1}{\gamma} - 2\partial_L \partial_T \gamma_{\underset{1}{LT}}\right)\\
&\quad \Rightarrow G_{\underset{1}{AB}} = R_{\underset{1}{AB}} - \tfrac{1}{2}\delta_{AB}\underset{1}{R} = \tfrac{1}{2}\left(\partial_L \partial_L \gamma_{\underset{1}{AB}} + \delta_{AB}\partial_L \partial_T \gamma_{\underset{1}{LT}} - \partial_L \partial_A \gamma_{\underset{1}{BL}} - \partial_L \partial_B \gamma_{\underset{1}{AL}}\right)
\end{aligned}$$

If we impose $\partial_L \gamma_{LA} = 0$ for the choice of the coordinate system, we obtain the following linearization of the fields equations $G_{\underset{1}{AB}} = 0$:

$$\partial_L \partial_L\, \gamma_{\underset{1}{AB}} = 0$$

These equations can be developed into Greek indices:

$$\begin{aligned}
&\partial_\epsilon \partial_\epsilon \gamma_{\underset{1}{\alpha\beta}} + \partial_{\bar\epsilon} \partial_{\bar\epsilon} \gamma_{\underset{1}{\alpha\beta}} = 0\\
&\partial_\epsilon \partial_\epsilon \gamma_{\underset{1}{\alpha\bar\beta}} + \partial_{\bar\epsilon} \partial_{\bar\epsilon} \gamma_{\underset{1}{\alpha\bar\beta}} = 0
\end{aligned}$$

At this point, we could say that the first equation is formally equivalent to the classical Poisson equation $\partial_l \partial_l \phi = 0$ of the gravitational field in the empty space. We could also say that the second equation is formally equivalent to the classical d'Alembert equation $\partial_\epsilon \partial_\epsilon A_l = 0$ of the electromagnetic field in the empty space. However, it is not clear which part of $\gamma_{\alpha\beta}$ plays the role of the gravitational potential ϕ and which part of $\gamma_{\alpha\bar\beta}$ plays the role of the electromagnetic potentials A_a. For this, we need to establish the classical equations of motions for charged mass points, which will be the subject of the next section.

11 The linearization of the fields equations: the "Einstein-Infeld-Hoffmann" (EIH) method

The so-called "Einstein-Infeld-Hoffmann" (EIH) method of linearization is more complicated than the classical one, but it leads to more physical results. For a detailed description of this method, we refer to [1, 2, 3].

The basic idea behind the EIH method is to take into account the fact that the time derivative of a quantity is small relatively to the quantity itself and to its spatial derivatives. If ϕ is such a quantity, it is assumed that $\partial_a\phi$ and $\partial_{\bar{a}}\phi$ are of the same order of magnitude than ϕ, and that $\partial_4\phi$ and $\partial_{\bar{4}}\phi$ are of the order $c^{-1}\phi$ (c being the speed of light).

As for the classical linearization method, the fields equations will be expanded by making use of the following quantities:

$$g_{AB} = \delta_{AB} + h_{AB} \ , \ \gamma_{AB} = h_{AB} - \tfrac{1}{2}\delta_{AB}h \Leftrightarrow h_{AB} = \gamma_{AB} - \tfrac{1}{6}\delta_{AB}\gamma$$

Following the EIH method, we shall assume that theses quantities can be expanded into the following series:

$$\begin{array}{lcl}
\gamma_{ab} = c^{-4}\underset{4}{\gamma_{ab}} + c^{-6}\underset{6}{\gamma_{ab}} + \cdots & , & \gamma_{a\bar{b}} = c^{-4}\underset{4}{\gamma_{a\bar{b}}} + c^{-6}\underset{6}{\gamma_{a\bar{b}}} + \cdots \\
\gamma_{a4} = c^{-3}\underset{3}{\gamma_{a4}} + c^{-5}\underset{5}{\gamma_{a4}} + \cdots & , & \gamma_{a\bar{4}} = c^{-3}\underset{3}{\gamma_{a\bar{4}}} + c^{-5}\underset{5}{\gamma_{a\bar{4}}} + \cdots \\
\gamma_{44} = c^{-2}\underset{2}{\gamma_{44}} + c^{-4}\underset{4}{\gamma_{44}} + \cdots & , & \gamma_{4\bar{4}} = c^{-2}\underset{2}{\gamma_{4\bar{4}}} + c^{-4}\underset{4}{\gamma_{4\bar{4}}} + \cdots \\
h_{ab} = c^{-2}\underset{2}{h_{ab}} + c^{-4}\underset{4}{h_{ab}} + \cdots & , & h_{a\bar{b}} = c^{-4}\underset{4}{h_{a\bar{b}}} + c^{-6}\underset{6}{h_{a\bar{b}}} + \cdots \\
h_{a4} = c^{-3}\underset{3}{h_{a4}} + c^{-5}\underset{5}{h_{a4}} + \cdots & , & h_{a\bar{4}} = c^{-3}\underset{3}{h_{a\bar{4}}} + c^{-5}\underset{5}{h_{a\bar{4}}} + \cdots \\
h_{44} = c^{-2}\underset{2}{h_{44}} + c^{-4}\underset{4}{h_{44}} + \cdots & , & h_{4\bar{4}} = c^{-2}\underset{2}{h_{4\bar{4}}} + c^{-4}\underset{4}{h_{4\bar{4}}} + \cdots
\end{array}$$

The different power expansion of the γ-series is justified by the fact that we shall impose $\partial_L\gamma_{LA} = 0 \Rightarrow \partial_l\gamma_{la} + \partial_4\gamma_{4a} + \partial_{\bar{l}}\gamma_{\bar{l}a} + \partial_{\bar{4}}\gamma_{\bar{4}a} = 0$ and $\partial_l\gamma_{l4} + \partial_4\gamma_{44} + \partial_{\bar{l}}\gamma_{\bar{l}4} + \partial_{\bar{4}}\gamma_{\bar{4}4} = 0$ for the choice of the coordinate system.

The power expansion of the h-series follows from the γ-series. The following components are needed:

$$\begin{array}{lcl}
\underset{2}{h_{ab}} = -\frac{1}{3}\delta_{ab}\underset{2}{\gamma_{44}} \Rightarrow \underset{2}{h_{ll}} = -\underset{2}{\gamma_{44}} & , & \underset{2}{h_{a\bar{b}}} = 0 \\
\underset{4}{h_{ab}} = \underset{4}{\gamma_{ab}} - \frac{1}{3}\delta_{ab}\underset{4}{\gamma_{ll}} - \frac{1}{3}\delta_{ab}\underset{4}{\gamma_{44}} \Rightarrow \underset{4}{h_{ll}} = -\underset{4}{\gamma_{44}} & , & \underset{4}{h_{a\bar{b}}} = \underset{4}{\gamma_{a\bar{b}}} \\
\underset{3}{h_{a4}} = \underset{3}{\gamma_{a4}} & , & \underset{3}{h_{a\bar{4}}} = \underset{3}{\gamma_{a\bar{4}}} \\
\underset{2}{h_{44}} = \frac{2}{3}\underset{2}{\gamma_{44}} & , & \underset{2}{h_{4\bar{4}}} = \underset{2}{\gamma_{4\bar{4}}} \\
\underset{4}{h_{44}} = \frac{2}{3}\underset{4}{\gamma_{44}} - \frac{1}{3}\underset{4}{\gamma_{ll}} & , & \underset{4}{h_{4\bar{4}}} = \underset{4}{\gamma_{4\bar{4}}}
\end{array}$$

We shall further impose that $\underset{2}{\gamma_{44}} = \underset{2}{\gamma_{\bar{4}\bar{4}}}$, $\underset{4}{\gamma_{44}} = \underset{4}{\gamma_{\bar{4}\bar{4}}}$ and $\underset{4}{\gamma_{ll}} = \underset{4}{\gamma_{\bar{l}\bar{l}}}$.

For $g_{AL}g^{LB} = \delta^B_A$, we have:

$$\underset{2}{h}^{ab} = -\underset{2}{h}_{ab} = \tfrac{1}{3}\delta_{ab}\underset{2}{\gamma}_{44} \quad , \quad \underset{2}{h}^{a\bar{b}} = -\underset{2}{h}_{a\bar{b}} = 0$$
$$\underset{3}{h}^{a4} = -\underset{3}{h}_{a4} = -\underset{3}{\gamma}_{a4} \quad , \quad \underset{3}{h}^{a\bar{4}} = -\underset{3}{h}_{a\bar{4}} = -\underset{3}{\gamma}_{a\bar{4}}$$
$$\underset{2}{h}^{44} = -\underset{2}{h}_{44} = -\tfrac{2}{3}\underset{2}{\gamma}_{44} \quad , \quad \underset{2}{h}^{4\bar{4}} = -\underset{2}{h}_{4\bar{4}} = -\underset{2}{\gamma}_{4\bar{4}}$$

By making use of these series, the fields equations can be expanded into successive approximations. As we shall see below, the first one leads to the mass and the charge conservation laws, and the second one leads to the equations of motion.

11.1 The first approximation: the mass and the charge conservation laws

For the components of the Christoffel symbols, we obtain the following expressions:

$$\underset{2}{\Gamma}^c_{ab} = \tfrac{1}{2}\partial_a\underset{2}{h}_{bc} + \tfrac{1}{2}\partial_b\underset{2}{h}_{ac} - \tfrac{1}{2}\partial_c\underset{2}{h}_{ab} \Rightarrow \underset{2}{\Gamma}^c_{ab} = -\tfrac{1}{6}\delta_{bc}\partial_a\underset{2}{\gamma}_{44} - \tfrac{1}{6}\delta_{ac}\partial_b\underset{2}{\gamma}_{44} + \tfrac{1}{6}\delta_{ab}\partial_c\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^4_{ab} = \tfrac{1}{2}\partial_a\underset{3}{h}_{b4} + \tfrac{1}{2}\partial_b\underset{3}{h}_{a4} - \tfrac{1}{2}\partial_4\underset{2}{h}_{ab} \Rightarrow \underset{3}{\Gamma}^4_{ab} = \tfrac{1}{2}\partial_a\underset{3}{\gamma}_{b4} + \tfrac{1}{2}\partial_b\underset{3}{\gamma}_{a4} + \tfrac{1}{6}\delta_{ab}\partial_4\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^c_{a4} = \tfrac{1}{2}\partial_a\underset{3}{h}_{4c} + \tfrac{1}{2}\partial_4\underset{2}{h}_{ac} - \tfrac{1}{2}\partial_c\underset{3}{h}_{a4} \Rightarrow \underset{3}{\Gamma}^c_{a4} = \tfrac{1}{2}\partial_a\underset{3}{\gamma}_{4c} - \tfrac{1}{6}\delta_{ac}\partial_4\underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_c\underset{3}{\gamma}_{a4}$$
$$\underset{2}{\Gamma}^4_{a4} = \tfrac{1}{2}\partial_a\underset{2}{h}_{44} \Rightarrow \underset{2}{\Gamma}^4_{a4} = \tfrac{1}{3}\partial_a\underset{2}{\gamma}_{44}$$
$$\underset{2}{\Gamma}^c_{44} = -\tfrac{1}{2}\partial_c\underset{2}{h}_{44} \Rightarrow \underset{2}{\Gamma}^c_{44} = -\tfrac{1}{3}\partial_c\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^4_{44} = \tfrac{1}{2}\partial_4\underset{2}{h}_{44} \Rightarrow \underset{3}{\Gamma}^4_{44} = \tfrac{1}{3}\partial_4\underset{2}{\gamma}_{44}$$
$$\underset{2}{\Gamma}^c_{a\bar{b}} = \tfrac{1}{2}\partial_{\bar{b}}\underset{2}{h}_{ac} \Rightarrow \underset{2}{\Gamma}^c_{a\bar{b}} = -\tfrac{1}{6}\delta_{ac}\partial_{\bar{b}}\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^4_{a\bar{b}} = \tfrac{1}{2}\partial_a\underset{3}{h}_{\bar{b}4} + \tfrac{1}{2}\partial_{\bar{b}}\underset{3}{h}_{a4} \Rightarrow \underset{3}{\Gamma}^4_{a\bar{b}} = \tfrac{1}{2}\partial_a\underset{3}{\gamma}_{\bar{b}4} + \tfrac{1}{2}\partial_{\bar{b}}\underset{3}{\gamma}_{a4}$$
$$\underset{3}{\Gamma}^c_{a\bar{4}} = \tfrac{1}{2}\partial_a\underset{3}{h}_{\bar{4}c} + \tfrac{1}{2}\partial_{\bar{4}}\underset{2}{h}_{ac} - \tfrac{1}{2}\partial_c\underset{3}{h}_{a\bar{4}} \Rightarrow \underset{3}{\Gamma}^c_{a\bar{4}} = \tfrac{1}{2}\partial_a\underset{3}{\gamma}_{\bar{4}c} - \tfrac{1}{6}\delta_{ac}\partial_{\bar{4}}\underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_c\underset{3}{\gamma}_{a\bar{4}}$$
$$\underset{2}{\Gamma}^4_{a\bar{4}} = \tfrac{1}{2}\partial_a\underset{2}{h}_{\bar{4}4} \Rightarrow \underset{2}{\Gamma}^4_{a\bar{4}} = \tfrac{1}{2}\partial_a\underset{2}{\gamma}_{\bar{4}4}$$
$$\underset{2}{\Gamma}^c_{4\bar{4}} = -\tfrac{1}{2}\partial_c\underset{2}{h}_{4\bar{4}} \Rightarrow \underset{2}{\Gamma}^c_{4\bar{4}} = -\tfrac{1}{2}\partial_c\underset{2}{\gamma}_{4\bar{4}}$$
$$\underset{3}{\Gamma}^4_{4\bar{4}} = \tfrac{1}{2}\partial_{\bar{4}}\underset{2}{h}_{44} \Rightarrow \underset{3}{\Gamma}^4_{4\bar{4}} = \tfrac{1}{3}\partial_{\bar{4}}\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^c_{4\bar{b}} = \tfrac{1}{2}\partial_{\bar{b}}\underset{3}{h}_{4c} - \tfrac{1}{2}\partial_c\underset{3}{h}_{4\bar{b}} \Rightarrow \underset{3}{\Gamma}^c_{4\bar{b}} = \tfrac{1}{2}\partial_{\bar{b}}\underset{3}{\gamma}_{4c} - \tfrac{1}{2}\partial_c\underset{3}{\gamma}_{4\bar{b}}$$
$$\underset{2}{\Gamma}^4_{4\bar{b}} = \tfrac{1}{2}\partial_{\bar{b}}\underset{2}{h}_{44} \Rightarrow \underset{2}{\Gamma}^4_{4\bar{b}} = \tfrac{1}{3}\partial_{\bar{b}}\underset{2}{\gamma}_{44}$$
$$\underset{2}{\Gamma}^{\bar{c}}_{ab} = -\tfrac{1}{2}\partial_{\bar{c}}\underset{2}{h}_{ab} \Rightarrow \underset{2}{\Gamma}^{\bar{c}}_{ab} = \tfrac{1}{6}\delta_{ab}\partial_{\bar{c}}\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^{\bar{4}}_{ab} = \tfrac{1}{2}\partial_a\underset{3}{h}_{b\bar{4}} + \tfrac{1}{2}\partial_b\underset{3}{h}_{a\bar{4}} - \tfrac{1}{2}\partial_{\bar{4}}\underset{2}{h}_{ab} \Rightarrow \underset{3}{\Gamma}^{\bar{4}}_{ab} = \tfrac{1}{2}\partial_a\underset{3}{\gamma}_{b\bar{4}} + \tfrac{1}{2}\partial_b\underset{3}{\gamma}_{a\bar{4}} + \tfrac{1}{6}\delta_{ab}\partial_{\bar{4}}\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^{\bar{c}}_{a4} = \tfrac{1}{2}\partial_a\underset{3}{h}_{4\bar{c}} - \tfrac{1}{2}\partial_{\bar{c}}\underset{3}{h}_{a4} \Rightarrow \underset{3}{\Gamma}^{\bar{c}}_{a4} = \tfrac{1}{2}\partial_a\underset{3}{\gamma}_{4\bar{c}} - \tfrac{1}{2}\partial_{\bar{c}}\underset{3}{\gamma}_{a4}$$
$$\underset{2}{\Gamma}^{\bar{4}}_{a4} = \tfrac{1}{2}\partial_a\underset{2}{h}_{4\bar{4}} \Rightarrow \underset{2}{\Gamma}^{\bar{4}}_{a4} = \tfrac{1}{2}\partial_a\underset{2}{\gamma}_{4\bar{4}}$$
$$\underset{2}{\Gamma}^{\bar{c}}_{44} = -\tfrac{1}{2}\partial_{\bar{c}}\underset{2}{h}_{44} \Rightarrow \underset{2}{\Gamma}^{\bar{c}}_{44} = -\tfrac{1}{3}\partial_{\bar{c}}\underset{2}{\gamma}_{44}$$
$$\underset{3}{\Gamma}^{\bar{4}}_{44} = \partial_4\underset{2}{h}_{4\bar{4}} - \tfrac{1}{2}\partial_{\bar{4}}\underset{2}{h}_{44} \Rightarrow \underset{3}{\Gamma}^{\bar{4}}_{44} = \partial_4\underset{2}{\gamma}_{4\bar{4}} - \tfrac{1}{3}\partial_{\bar{4}}\underset{2}{\gamma}_{44}$$

For the components of the Ricci curvature tensor, we obtain the following expressions:

$$\underset{2}{R}_{ab} = \partial_a \underset{2}{\Gamma}^{l}_{bl} + \partial_a \underset{2}{\Gamma}^{4}_{b4} + \partial_a \underset{2}{\Gamma}^{\bar l}_{b\bar l} + \partial_a \underset{2}{\Gamma}^{\bar 4}_{b\bar 4} - \partial_l \underset{2}{\Gamma}^{l}_{ab} - \partial_{\bar l} \underset{2}{\Gamma}^{\bar l}_{ab} \Rightarrow \underset{2}{R}_{ab} = -\tfrac{1}{6}\delta_{ab}\left(\partial_l\partial_l \underset{2}{\gamma}_{44} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{44}\right)$$

$$\underset{3}{R}_{a4} = \partial_a \underset{3}{\Gamma}^{l}_{4l} + \partial_a \underset{3}{\Gamma}^{4}_{44} + \partial_a \underset{3}{\Gamma}^{\bar l}_{4\bar l} + \partial_a \underset{3}{\Gamma}^{\bar 4}_{4\bar 4} - \partial_l \underset{3}{\Gamma}^{l}_{a4} - \partial_4 \underset{2}{\Gamma}^{4}_{a4} - \partial_{\bar l} \underset{3}{\Gamma}^{\bar l}_{a4} - \partial_{\bar 4} \underset{2}{\Gamma}^{\bar 4}_{a4}$$

$$\Rightarrow \underset{3}{R}_{a4} = -\tfrac{1}{2}\partial_a\partial_4 \underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_a\partial_{\bar 4}\underset{2}{\gamma}_{4\bar 4} + \tfrac{1}{2}\partial_l\left(\partial_l \underset{3}{\gamma}_{a4} - \partial_a \underset{3}{\gamma}_{l4}\right) + \tfrac{1}{2}\partial_{\bar l}\left(\partial_{\bar l}\underset{3}{\gamma}_{a4} - \partial_a \underset{3}{\gamma}_{\bar l 4}\right)$$

$$\underset{2}{R}_{44} = -\partial_l \underset{2}{\Gamma}^{l}_{44} - \partial_{\bar l}\underset{2}{\Gamma}^{\bar l}_{44} \Rightarrow \underset{2}{R}_{44} = \tfrac{1}{3}\left(\partial_l\partial_l \underset{2}{\gamma}_{44} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{44}\right)$$

$$\underset{2}{R}_{a\bar b} = \partial_a \underset{2}{\Gamma}^{l}_{\bar b l} + \partial_a \underset{2}{\Gamma}^{4}_{\bar b 4} + \partial_a \underset{2}{\Gamma}^{\bar l}_{\bar b\bar l} + \partial_a \underset{2}{\Gamma}^{\bar 4}_{\bar b\bar 4} - \partial_l \underset{2}{\Gamma}^{l}_{a\bar b} - \partial_{\bar l} \underset{2}{\Gamma}^{\bar l}_{a\bar b} \Rightarrow \underset{2}{R}_{a\bar b} = 0$$

$$\underset{3}{R}_{a\bar 4} = \partial_a \underset{3}{\Gamma}^{l}_{\bar 4 l} + \partial_a \underset{3}{\Gamma}^{4}_{\bar 4 4} + \partial_a \underset{3}{\Gamma}^{\bar l}_{\bar 4\bar l} + \partial_a \underset{3}{\Gamma}^{\bar 4}_{\bar 4\bar 4} - \partial_l \underset{3}{\Gamma}^{l}_{a\bar 4} - \partial_4 \underset{2}{\Gamma}^{4}_{a\bar 4} - \partial_{\bar l} \underset{3}{\Gamma}^{\bar l}_{a\bar 4} - \partial_{\bar 4} \underset{2}{\Gamma}^{\bar 4}_{a\bar 4}$$

$$\Rightarrow \underset{3}{R}_{a\bar 4} = -\tfrac{1}{2}\partial_a\partial_{\bar 4} \underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_a\partial_{4}\underset{2}{\gamma}_{4\bar 4} + \tfrac{1}{2}\partial_l\left(\partial_l \underset{3}{\gamma}_{a\bar 4} - \partial_a \underset{3}{\gamma}_{l\bar 4}\right) + \tfrac{1}{2}\partial_{\bar l}\left(\partial_{\bar l}\underset{3}{\gamma}_{a\bar 4} - \partial_a \underset{3}{\gamma}_{\bar l\bar 4}\right)$$

$$\underset{2}{R}_{4\bar 4} = -\partial_l \underset{2}{\Gamma}^{l}_{4\bar 4} - \partial_{\bar l}\underset{2}{\Gamma}^{\bar l}_{4\bar 4} \Rightarrow \underset{2}{R}_{4\bar 4} = \tfrac{1}{2}\left(\partial_l\partial_l \underset{2}{\gamma}_{4\bar 4} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{4\bar 4}\right)$$

$$\underset{2}{R} = \underset{2}{R}_{ll} + \underset{2}{R}_{44} + \underset{2}{R}_{\bar l\bar l} + \underset{2}{R}_{\bar 4\bar 4} \Rightarrow \underset{2}{R} = -\tfrac{1}{3}\left(\partial_l\partial_l \underset{2}{\gamma}_{44} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{44}\right)$$

The fields equations of the first approximation become:

$$\left.\begin{array}{l}
\underset{2}{G}_{ab} = 0 \\
\underset{3}{G}_{a4} = -\tfrac{1}{2}\partial_a\partial_4 \underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_a\partial_{\bar 4}\underset{2}{\gamma}_{4\bar 4} + \tfrac{1}{2}\partial_l\left(\partial_l \underset{3}{\gamma}_{a4} - \partial_a \underset{3}{\gamma}_{l4}\right) + \tfrac{1}{2}\partial_{\bar l}\left(\partial_{\bar l}\underset{3}{\gamma}_{a4} - \partial_a \underset{3}{\gamma}_{\bar l 4}\right) = 0 \\
\underset{2}{G}_{44} = \tfrac{1}{2}\left(\partial_l\partial_l \underset{2}{\gamma}_{44} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{44}\right) = 0 \\
\underset{2}{G}_{a\bar b} = 0 \\
\underset{3}{G}_{a\bar 4} = -\tfrac{1}{2}\partial_a\partial_{\bar 4} \underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_a\partial_{4}\underset{2}{\gamma}_{4\bar 4} + \tfrac{1}{2}\partial_l\left(\partial_l \underset{3}{\gamma}_{a\bar 4} - \partial_a \underset{3}{\gamma}_{l\bar 4}\right) + \tfrac{1}{2}\partial_{\bar l}\left(\partial_{\bar l}\underset{3}{\gamma}_{a\bar 4} - \partial_a \underset{3}{\gamma}_{\bar l\bar 4}\right) = 0 \\
\underset{2}{G}_{4\bar 4} = \tfrac{1}{2}\left(\partial_l\partial_l \underset{2}{\gamma}_{4\bar 4} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{4\bar 4}\right) = 0
\end{array}\right| \tag{7}$$

We shall impose the following conditions for the choice of the coordinate system:

$$\partial_L\gamma_{LA} = 0 \Rightarrow \partial_l\gamma_{la} + \partial_4\gamma_{4a} + \partial_{\bar l}\gamma_{\bar l a} + \partial_{\bar 4}\gamma_{\bar 4 a} = 0 \text{ and } \partial_l\gamma_{l4} + \partial_4\gamma_{44} + \partial_{\bar l}\gamma_{\bar l 4} + \partial_{\bar 4}\gamma_{\bar 4 4} = 0 \tag{8}$$

The fields equations of the first approximation become:

$$\left.\begin{array}{lcl}
\underset{2}{G}_{ab} = 0 & , & \underset{2}{G}_{a\bar b} = 0 \\
\underset{3}{G}_{a4} = \tfrac{1}{2}\left(\partial_l\partial_l \underset{3}{\gamma}_{a4} + \partial_{\bar l}\partial_{\bar l}\underset{3}{\gamma}_{a4}\right) = 0 & , & \underset{3}{G}_{a\bar 4} = \tfrac{1}{2}\left(\partial_l\partial_l \underset{3}{\gamma}_{a\bar 4} + \partial_{\bar l}\partial_{\bar l}\underset{3}{\gamma}_{a\bar 4}\right) = 0 \\
\underset{2}{G}_{44} = \tfrac{1}{2}\left(\partial_l\partial_l \underset{2}{\gamma}_{44} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{44}\right) = 0 & , & \underset{2}{G}_{4\bar 4} = \tfrac{1}{2}\left(\partial_l\partial_l \underset{2}{\gamma}_{4\bar 4} + \partial_{\bar l}\partial_{\bar l}\underset{2}{\gamma}_{4\bar 4}\right) = 0 \\
\partial_l \underset{4}{\gamma}_{la} + \partial_4 \underset{3}{\gamma}_{4a} + \partial_{\bar l}\underset{4}{\gamma}_{\bar l a} + \partial_{\bar 4}\underset{3}{\gamma}_{\bar 4 a} = 0 & , & \partial_l \underset{3}{\gamma}_{l4} + \partial_4 \underset{2}{\gamma}_{44} + \partial_{\bar l}\underset{3}{\gamma}_{\bar l 4} + \partial_{\bar 4}\underset{2}{\gamma}_{\bar 4 4} = 0
\end{array}\right| \tag{9}$$

The mathematical world of classical physics is based on a 4-dimensional spacetime. In order to take this fact into account, we shall further restrict the equations (9) so that they hold separately in each of the two 4-dimensional spacetime being considered:

$$\left.\begin{array}{lcl}
\partial_l\partial_l\underset{2}{\gamma}_{44}\left(z^{\alpha}\right)=\partial_l\partial_l\underset{2}{\gamma}_{\bar{4}\bar{4}}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\left(z^{\bar{\alpha}}\right)=\partial_{\bar{l}}\partial_{\bar{l}}\underset{2}{\gamma}_{\bar{4}\bar{4}}\left(z^{\bar{\alpha}}\right)=0 \\
\partial_l\partial_l\underset{3}{\gamma}_{a4}\left(z^{\alpha}\right)=\partial_l\partial_l\underset{3}{\gamma}_{\bar{a}\bar{4}}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\partial_{\bar{l}}\underset{3}{\gamma}_{a4}\left(z^{\bar{\alpha}}\right)=\partial_{\bar{l}}\partial_{\bar{l}}\underset{3}{\gamma}_{\bar{a}\bar{4}}\left(z^{\bar{\alpha}}\right)=0 \\
\partial_l\partial_l\underset{2}{\gamma}_{4\bar{4}}\left(z^{\alpha}\right)=\partial_l\partial_l\underset{2}{\gamma}_{\bar{4}4}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\left(z^{\bar{\alpha}}\right)=\partial_{\bar{l}}\partial_{\bar{l}}\underset{2}{\gamma}_{\bar{4}4}\left(z^{\bar{\alpha}}\right)=0 \\
\partial_l\partial_l\underset{3}{\gamma}_{a\bar{4}}\left(z^{\alpha}\right)=\partial_l\partial_l\underset{3}{\gamma}_{\bar{a}4}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\partial_{\bar{l}}\underset{3}{\gamma}_{a\bar{4}}\left(z^{\bar{\alpha}}\right)=\partial_{\bar{l}}\partial_{\bar{l}}\underset{3}{\gamma}_{\bar{a}4}\left(z^{\bar{\alpha}}\right)=0 \\
\partial_l\underset{4}{\gamma}_{la}\left(z^{\alpha}\right)+\partial_4\underset{3}{\gamma}_{4a}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\underset{4}{\gamma}_{\bar{l}\bar{a}}\left(z^{\bar{\alpha}}\right)+\partial_{\bar{4}}\underset{3}{\gamma}_{\bar{4}\bar{a}}\left(z^{\bar{\alpha}}\right)=0 \\
\partial_l\underset{3}{\gamma}_{l4}\left(z^{\alpha}\right)+\partial_4\underset{2}{\gamma}_{44}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\underset{3}{\gamma}_{\bar{l}\bar{4}}\left(z^{\bar{\alpha}}\right)+\partial_{\bar{4}}\underset{2}{\gamma}_{\bar{4}\bar{4}}\left(z^{\bar{\alpha}}\right)=0 \\
\partial_l\underset{4}{\gamma}_{l\bar{a}}\left(z^{\alpha}\right)+\partial_4\underset{3}{\gamma}_{4\bar{a}}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\underset{4}{\gamma}_{\bar{l}a}\left(z^{\bar{\alpha}}\right)+\partial_{\bar{4}}\underset{3}{\gamma}_{\bar{4}a}\left(z^{\bar{\alpha}}\right)=0 \\
\partial_l\underset{3}{\gamma}_{l\bar{4}}\left(z^{\alpha}\right)+\partial_4\underset{2}{\gamma}_{4\bar{4}}\left(z^{\alpha}\right)=0 & , & \partial_{\bar{l}}\underset{3}{\gamma}_{\bar{l}4}\left(z^{\bar{\alpha}}\right)+\partial_{\bar{4}}\underset{2}{\gamma}_{\bar{4}4}\left(z^{\bar{\alpha}}\right)=0
\end{array}\right| \qquad (10)$$

It is clear that the solutions to the equations (10) are also solutions to the equations (9) if they have the following form:

$$\gamma_{AB}=\gamma_{AB}\left(z^{\alpha},z^{\bar{\alpha}}\right)=\gamma_{AB}\left(z^{\alpha}\right)+\gamma_{AB}\left(z^{\bar{\alpha}}\right)\Rightarrow\gamma_{\epsilon\theta}=\gamma_{\bar{\epsilon}\bar{\theta}}\ ,\ \gamma_{\epsilon\bar{\theta}}=\gamma_{\bar{\epsilon}\theta}\text{ and }\partial_{\epsilon}\partial_{\bar{\theta}}\gamma_{AB}=0 \qquad (11)$$

At this point, we can understand why the Lorentz transformation must be linear in classical physics. For the metric transforms according to (2), we conclude that the only coordinate transformations that preserve the form of (11) are the linear coordinate transformations. From the theory of relativity, we know that the Lorentz transformation is the only linear coordinate transformation which preserves the form of the Maxwell equations. The conclusion that the Lorentz transformation must be linear provides the explanation to the fact that certain coordinate systems (the *inertial systems*) are privileged in classical physics.

As the solutions of equations (10), we shall choose the following expressions:

$$\left.\begin{array}{l}
\underset{2}{\gamma}_{44} = -\sum_{k=1}^{N} \overset{k}{\mu} \left(\frac{1}{\overset{k}{r}} + \frac{1}{\overset{k}{\bar{r}}} \right) \quad , \quad \underset{2}{\gamma}_{4\bar{4}} = +\sum_{k=1}^{N} \overset{k}{\rho} \left(\frac{1}{\overset{k}{r}} + \frac{1}{\overset{k}{\bar{r}}} \right) \\
\underset{3}{\gamma}_{a4} = -\sum_{k=1}^{N} \overset{k}{\mu} \left(\frac{\partial_4 \overset{k}{y}^a}{\overset{k}{r}} + \frac{\partial_{\bar{4}} \overset{k}{\bar{y}}^a}{\overset{k}{\bar{r}}} \right) \quad , \quad \underset{3}{\gamma}_{a\bar{4}} = +\sum_{k=1}^{N} \overset{k}{\rho} \left(\frac{\partial_4 \overset{k}{y}^a}{\overset{k}{r}} + \frac{\partial_{\bar{4}} \overset{k}{\bar{y}}^a}{\overset{k}{\bar{r}}} \right) \\
\underset{4}{\gamma}_{ab} = -\sum_{k=1}^{N} \overset{k}{\mu} \left(\frac{\partial_4 \overset{k}{y}^a \, \partial_4 \overset{k}{y}^b}{\overset{k}{r}} + \frac{\partial_{\bar{4}} \overset{k}{\bar{y}}^a \, \partial_{\bar{4}} \overset{k}{\bar{y}}^b}{\overset{k}{\bar{r}}} \right) \quad , \quad \underset{4}{\gamma}_{a\bar{b}} = +\sum_{k=1}^{N} \overset{k}{\rho} \left(\frac{\partial_4 \overset{k}{y}^a \, \partial_4 \overset{k}{y}^b}{\overset{k}{r}} + \frac{\partial_{\bar{4}} \overset{k}{\bar{y}}^a \, \partial_{\bar{4}} \overset{k}{\bar{y}}^b}{\overset{k}{\bar{r}}} \right) \\
\text{with } \overset{k}{r} = \sqrt{\left(z^l - \overset{k}{y}^l\right)\left(z^l - \overset{k}{y}^l\right)} \Rightarrow \partial_a \overset{k}{r} = \frac{\left(z^a - \overset{k}{y}^a\right)}{\overset{k}{r}} \text{ and } \partial_4 \overset{k}{r} = -\frac{\partial_4 \overset{k}{y}^l \left(z^l - \overset{k}{y}^l\right)}{\overset{k}{r}} \\
\text{and } \overset{k}{\bar{r}} = \sqrt{\left(z^{\bar{l}} - \overset{k}{y}^{\bar{l}}\right)\left(z^{\bar{l}} - \overset{k}{y}^{\bar{l}}\right)} \Rightarrow \partial_{\bar{a}} \overset{k}{\bar{r}} = \frac{\left(z^{\bar{a}} - \overset{k}{y}^{\bar{a}}\right)}{\overset{k}{\bar{r}}} \text{ and } \partial_{\bar{4}} \overset{k}{\bar{r}} = -\frac{\partial_{\bar{4}} \overset{k}{y}^{\bar{l}} \left(z^{\bar{l}} - \overset{k}{y}^{\bar{l}}\right)}{\overset{k}{\bar{r}}}
\end{array}\right| \tag{12}$$

These solutions represents N charged mass points. The $3N$ functions $\overset{k}{y}^l$ of z^4 determine the locations of the N charged mass points at every time z^4 .

We shall now prove that the masses $\overset{k}{\mu}$ and the charges $\overset{k}{\rho}$ are constant. If we take into account the results of (10) and (11) to rewrite the fields equations (7), we obtain:

$$\underset{3}{G}_{a4} = -\tfrac{1}{2}\partial_a \partial_4 \underset{2}{\gamma}_{44} + \tfrac{1}{2}\partial_l \left(\partial_l \underset{3}{\gamma}_{a4} - \partial_a \underset{3}{\gamma}_{l4} \right) = 0$$

$$\underset{3}{G}_{a\bar{4}} = -\tfrac{1}{2}\partial_a \partial_4 \underset{2}{\gamma}_{4\bar{4}} + \tfrac{1}{2}\partial_l \left(\partial_l \underset{3}{\gamma}_{a\bar{4}} - \partial_a \underset{3}{\gamma}_{l\bar{4}} \right) = 0$$

Each of these equations contain the divergence of a skewsymmetric expression which is equivalent to a curl. According to the Stockes's theorem, the integral of a curl taken over a closed surface vanishes. If we apply this result to a closed surface which encloses the *pth* charged mass point, we obtain (with η_a being the cosine of the angle between the z^a coordinate and the normal of the surface element dS):

$$\oint_S \underset{3}{G}_{a4} \eta_a dS = 0 \Rightarrow -\tfrac{1}{2}\oint_S \partial_a \partial_4 \underset{2}{\gamma}_{44} \eta_a dS + \overbrace{\tfrac{1}{2}\oint_S \partial_l \left(\partial_l \underset{3}{\gamma}_{a4} - \partial_a \underset{3}{\gamma}_{l4} \right) \eta_a dS}^{=0} = 0 \Rightarrow \partial_4 \oint_S \partial_a \underset{2}{\gamma}_{44} \eta_a dS = 0$$

$$\Rightarrow \frac{d \overset{p}{\mu}}{dz^4} = 0 \Rightarrow \overset{p}{\mu} \text{ is constant.}$$

$$\oint_S \underset{3}{G}_{a\bar{4}} \eta_a dS = 0 \Rightarrow -\tfrac{1}{2}\oint_S \partial_a \partial_4 \underset{2}{\gamma}_{4\bar{4}} \eta_a dS + \overbrace{\tfrac{1}{2}\oint_S \partial_l \left(\partial_l \underset{3}{\gamma}_{a\bar{4}} - \partial_a \underset{3}{\gamma}_{l\bar{4}} \right) \eta_a dS}^{=0} = 0 \Rightarrow \partial_4 \oint_S \partial_a \underset{2}{\gamma}_{4\bar{4}} \eta_a dS = 0$$

$$\Rightarrow \frac{d \overset{p}{\rho}}{dz^4} = 0 \Rightarrow \overset{p}{\rho} \text{ is constant.}$$

11.2 The second approximation: the equations of motion

For the components of the Christoffel symbols, the following expressions are needed:

$$
\begin{aligned}
&\underset{4}{\Gamma}{}^{c}_{ab}= \tfrac{1}{2}\partial_a \underset{4}{h}_{bc} + \tfrac{1}{2}\partial_b \underset{4}{h}_{ac} - \tfrac{1}{2}\partial_c \underset{4}{h}_{ab} - \tfrac{1}{2}\underset{2}{h}_{cl}\partial_a \underset{2}{h}_{bl} - \tfrac{1}{2}\underset{2}{h}_{cl}\partial_b \underset{2}{h}_{al} + \tfrac{1}{2}\underset{2}{h}_{cl}\partial_l \underset{2}{h}_{ab}\\
&\quad\Rightarrow \underset{4}{\Gamma}{}^{c}_{ab} =\tfrac{1}{2}\partial_a \underset{4}{\gamma}_{bc} - \tfrac{1}{6}\delta_{bc}\partial_a \underset{4}{\gamma}_{ll} - \tfrac{1}{6}\delta_{bc}\partial_a \underset{4}{\gamma}_{44} + \tfrac{1}{2}\partial_b \underset{4}{\gamma}_{ac} - \tfrac{1}{6}\delta_{ac}\partial_b \underset{4}{\gamma}_{ll} - \tfrac{1}{6}\delta_{ac}\partial_b \underset{4}{\gamma}_{44} - \tfrac{1}{2}\partial_c \underset{4}{\gamma}_{ab}\\
&\qquad\qquad -\tfrac{1}{6}\delta_{ab}\partial_c \underset{4}{\gamma}_{ll} - \tfrac{1}{6}\delta_{ab}\partial_c \underset{4}{\gamma}_{44} - \tfrac{1}{18}\delta_{bc}\underset{2}{\gamma}_{44}\partial_a \underset{2}{\gamma}_{44} - \tfrac{1}{18}\delta_{ac}\underset{2}{\gamma}_{44}\partial_b \underset{2}{\gamma}_{44} + \tfrac{1}{18}\delta_{ab}\underset{2}{\gamma}_{44}\partial_c \underset{2}{\gamma}_{44}\\
&\underset{4}{\Gamma}{}^{4}_{a4}= \tfrac{1}{2}\partial_a \underset{4}{h}_{44} - \tfrac{1}{2}\underset{2}{h}_{44}\partial_a \underset{2}{h}_{44} - \tfrac{1}{2}\underset{2}{h}_{4\bar{4}}\partial_a \underset{2}{h}_{4\bar{4}}\\
&\quad\Rightarrow \underset{4}{\Gamma}{}^{4}_{a4} = \tfrac{1}{3}\partial_a \underset{4}{\gamma}_{44} - \tfrac{1}{6}\partial_a \underset{4}{\gamma}_{ll} - \tfrac{2}{9}\underset{2}{\gamma}_{44}\partial_a \underset{2}{\gamma}_{44} - \tfrac{1}{2}\underset{2}{\gamma}_{4\bar{4}}\partial_a \underset{2}{\gamma}_{4\bar{4}}\\
&\underset{4}{\Gamma}{}^{c}_{a\bar{b}}= \tfrac{1}{2}\partial_a \underset{4}{h}_{\bar{b}c} + \tfrac{1}{2}\partial_{\bar{b}} \underset{4}{h}_{ac} - \tfrac{1}{2}\partial_c \underset{4}{h}_{a\bar{b}} - \tfrac{1}{2}\underset{2}{h}_{cl}\partial_{\bar{b}} \underset{2}{h}_{al}\\
&\quad\Rightarrow \underset{4}{\Gamma}{}^{c}_{a\bar{b}} = \tfrac{1}{2}\partial_a \underset{4}{\gamma}_{\bar{b}c} + \tfrac{1}{2}\partial_{\bar{b}} \underset{4}{\gamma}_{ac} - \tfrac{1}{6}\delta_{ac}\partial_{\bar{b}} \underset{4}{\gamma}_{ll} - \tfrac{1}{6}\delta_{ac}\partial_{\bar{b}} \underset{4}{\gamma}_{44} - \tfrac{1}{2}\partial_c \underset{4}{\gamma}_{a\bar{b}} - \tfrac{1}{18}\delta_{ac}\underset{2}{\gamma}_{44}\partial_{\bar{b}} \underset{2}{\gamma}_{4\bar{4}}\\
&\underset{4}{\Gamma}{}^{4}_{4\bar{b}}= \tfrac{1}{2}\partial_{\bar{b}} \underset{4}{h}_{44} - \tfrac{1}{2}\underset{2}{h}_{44}\partial_{\bar{b}} \underset{2}{h}_{44} - \tfrac{1}{2}\underset{2}{h}_{4\bar{4}}\partial_{\bar{b}} \underset{2}{h}_{4\bar{4}}\\
&\quad\Rightarrow \underset{4}{\Gamma}{}^{4}_{4\bar{b}} = \tfrac{1}{3}\partial_{\bar{b}} \underset{4}{\gamma}_{44} - \tfrac{1}{6}\partial_{\bar{b}} \underset{4}{\gamma}_{ll} - \tfrac{2}{9}\underset{2}{\gamma}_{44}\partial_{\bar{b}} \underset{2}{\gamma}_{44} - \tfrac{1}{2}\underset{2}{\gamma}_{4\bar{4}}\partial_{\bar{b}} \underset{2}{\gamma}_{4\bar{4}}\\
&\underset{4}{\Gamma}{}^{\bar{c}}_{ab}= \tfrac{1}{2}\partial_a \underset{4}{h}_{b\bar{c}} + \tfrac{1}{2}\partial_b \underset{4}{h}_{a\bar{c}} - \tfrac{1}{2}\partial_{\bar{c}} \underset{4}{h}_{ab} + \tfrac{1}{2}\underset{2}{h}_{cl}\partial_l \underset{2}{h}_{ab}\\
&\quad\Rightarrow \underset{4}{\Gamma}{}^{\bar{c}}_{ab} = \tfrac{1}{2}\partial_a \underset{4}{\gamma}_{b\bar{c}} + \tfrac{1}{2}\partial_b \underset{4}{\gamma}_{a\bar{c}} - \tfrac{1}{2}\partial_{\bar{c}} \underset{4}{\gamma}_{ab} + \tfrac{1}{6}\delta_{ab}\partial_{\bar{c}} \underset{4}{\gamma}_{ll} + \tfrac{1}{6}\delta_{ab}\partial_{\bar{c}} \underset{4}{\gamma}_{44} + \tfrac{1}{18}\delta_{ab}\underset{2}{\gamma}_{44}\partial_{\bar{c}} \underset{2}{\gamma}_{44}
\end{aligned}
$$

For the components of the Ricci curvature tensor, the following expressions are needed:

$$
\begin{aligned}
&\underset{4}{R}_{ab}=\partial_a \underset{4}{\Gamma}{}^{l}_{bl} + \partial_a \underset{4}{\Gamma}{}^{4}_{b4} + \partial_a \underset{4}{\Gamma}{}^{\bar{l}}_{b\bar{l}} + \partial_a \underset{4}{\Gamma}{}^{\bar{4}}_{b\bar{4}} - \partial_l \underset{4}{\Gamma}{}^{l}_{ab} - \partial_4 \underset{3}{\Gamma}{}^{4}_{ab} - \partial_{\bar{l}} \underset{4}{\Gamma}{}^{\bar{l}}_{ab} - \partial_{\bar{4}} \underset{3}{\Gamma}{}^{\bar{4}}_{ab} + \underset{2}{\Gamma}{}^{l}_{at}\underset{2}{\Gamma}{}^{t}_{lb} + \underset{2}{\Gamma}{}^{l}_{a\bar{t}}\underset{2}{\Gamma}{}^{\bar{t}}_{lb}\\
&\qquad +\underset{2}{\Gamma}{}^{4}_{a4}\underset{2}{\Gamma}{}^{4}_{4b} + \underset{2}{\Gamma}{}^{4}_{a\bar{4}}\underset{2}{\Gamma}{}^{\bar{4}}_{4b} + \underset{2}{\Gamma}{}^{\bar{l}}_{at}\underset{2}{\Gamma}{}^{t}_{\bar{l}b} + \underset{2}{\Gamma}{}^{\bar{l}}_{a\bar{t}}\underset{2}{\Gamma}{}^{\bar{t}}_{\bar{l}b} + \underset{2}{\Gamma}{}^{\bar{4}}_{a4}\underset{2}{\Gamma}{}^{4}_{\bar{4}b} + \underset{2}{\Gamma}{}^{\bar{4}}_{a\bar{4}}\underset{2}{\Gamma}{}^{\bar{4}}_{\bar{4}b} - \underset{2}{\Gamma}{}^{t}_{ab}\underset{2}{\Gamma}{}^{l}_{tl} - \underset{2}{\Gamma}{}^{\bar{t}}_{ab}\underset{2}{\Gamma}{}^{l}_{\bar{t}l} - \underset{2}{\Gamma}{}^{t}_{ab}\underset{2}{\Gamma}{}^{4}_{t4} - \underset{2}{\Gamma}{}^{\bar{t}}_{ab}\underset{2}{\Gamma}{}^{4}_{\bar{t}4}\\
&\qquad -\underset{2}{\Gamma}{}^{t}_{ab}\underset{2}{\Gamma}{}^{\bar{l}}_{t\bar{l}} - \underset{2}{\Gamma}{}^{\bar{t}}_{ab}\underset{2}{\Gamma}{}^{\bar{l}}_{\bar{t}\bar{l}} - \underset{2}{\Gamma}{}^{t}_{ab}\underset{2}{\Gamma}{}^{\bar{4}}_{t\bar{4}} - \underset{2}{\Gamma}{}^{\bar{t}}_{ab}\underset{2}{\Gamma}{}^{\bar{4}}_{\bar{t}\bar{4}}\\
&\quad\Rightarrow \underset{4}{R}_{ab} =-\tfrac{1}{2}\partial_a\partial_l \underset{4}{\gamma}_{bl} - \tfrac{1}{2}\partial_b\partial_l \underset{4}{\gamma}_{al} + \tfrac{1}{2}\partial_l\partial_l \underset{4}{\gamma}_{ab} + \tfrac{1}{2}\partial_{\bar{l}}\partial_{\bar{l}} \underset{4}{\gamma}_{ab}\\
&\qquad\qquad -\tfrac{1}{6}\delta ab\partial_l\partial_l \underset{4}{\gamma}_{tt} - \tfrac{1}{6}\delta ab\partial_{\bar{l}}\partial_{\bar{l}} \underset{4}{\gamma}_{tt} - \tfrac{1}{6}\delta ab\partial_l\partial_l \underset{4}{\gamma}_{44} - \tfrac{1}{6}\delta ab\partial_{\bar{l}}\partial_{\bar{l}} \underset{4}{\gamma}_{44}\\
&\qquad\qquad -\tfrac{1}{2}\partial_a\partial_{\bar{l}} \underset{4}{\gamma}_{b\bar{l}} - \tfrac{1}{2}\partial_b\partial_{\bar{l}} \underset{4}{\gamma}_{a\bar{l}}\\
&\qquad\qquad -\tfrac{1}{6}\delta_{ab}\partial_4\partial_4 \underset{2}{\gamma}_{44} - \tfrac{1}{6}\delta_{ab}\partial_{\bar{4}}\partial_{\bar{4}} \underset{2}{\gamma}_{44} - \tfrac{1}{3}\partial_a \underset{2}{\gamma}_{44}\partial_b \underset{2}{\gamma}_{44} - \tfrac{2}{3}\underset{2}{\gamma}_{44}\partial_a\partial_b \underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_a \underset{2}{\gamma}_{4\bar{4}}\partial_b \underset{2}{\gamma}_{4\bar{4}} - \underset{2}{\gamma}_{4\bar{4}}\partial_a\partial_b \underset{2}{\gamma}_{4\bar{4}}\\
&\qquad\qquad -\tfrac{1}{18}\delta_{ab}\partial_l \underset{2}{\gamma}_{44}\partial_l \underset{2}{\gamma}_{44} - \tfrac{1}{18}\delta_{ab}\partial_{\bar{l}} \underset{2}{\gamma}_{44}\partial_{\bar{l}} \underset{2}{\gamma}_{44} - \tfrac{1}{18}\delta_{ab}\underset{2}{\gamma}_{44}\partial_l\partial_l \underset{2}{\gamma}_{44} - \tfrac{1}{18}\delta_{ab}\underset{2}{\gamma}_{44}\partial_{\bar{l}}\partial_{\bar{l}} \underset{2}{\gamma}_{44}\\
&\qquad\qquad -\tfrac{1}{2}\partial_a\partial_4 \underset{3}{\gamma}_{b4} - \tfrac{1}{2}\partial_b\partial_4 \underset{3}{\gamma}_{a4}\\
&\qquad\qquad -\tfrac{1}{2}\partial_a\partial_{\bar{4}} \underset{3}{\gamma}_{b\bar{4}} - \tfrac{1}{2}\partial_b\partial_{\bar{4}} \underset{3}{\gamma}_{a\bar{4}}
\end{aligned}
$$

$$
\begin{aligned}
&\underset{4}{R}_{44}=\partial_4 \underset{3}{\Gamma}{}^{l}_{4l} + \partial_4 \underset{3}{\Gamma}{}^{\bar{l}}_{4\bar{l}} + \partial_4 \underset{3}{\Gamma}{}^{\bar{4}}_{4\bar{4}} - \partial_l \underset{4}{\Gamma}{}^{l}_{44} - \partial_{\bar{l}} \underset{4}{\Gamma}{}^{\bar{l}}_{44} - \partial_{\bar{4}} \underset{3}{\Gamma}{}^{\bar{4}}_{44} + \underset{2}{\Gamma}{}^{l}_{4\bar{4}}\underset{2}{\Gamma}{}^{\bar{4}}_{l4} + \underset{2}{\Gamma}{}^{4}_{4t}\underset{2}{\Gamma}{}^{t}_{44} + \underset{2}{\Gamma}{}^{4}_{4\bar{t}}\underset{2}{\Gamma}{}^{\bar{t}}_{44} + \underset{2}{\Gamma}{}^{\bar{l}}_{4\bar{4}}\underset{2}{\Gamma}{}^{\bar{4}}_{\bar{l}4}\\
&\qquad +\underset{2}{\Gamma}{}^{\bar{4}}_{4t}\underset{2}{\Gamma}{}^{t}_{\bar{4}4} + \underset{2}{\Gamma}{}^{\bar{4}}_{4\bar{t}}\underset{2}{\Gamma}{}^{\bar{t}}_{\bar{4}4} - \underset{2}{\Gamma}{}^{t}_{44}\underset{2}{\Gamma}{}^{l}_{tl} - \underset{2}{\Gamma}{}^{\bar{t}}_{44}\underset{2}{\Gamma}{}^{l}_{\bar{t}l} - \underset{2}{\Gamma}{}^{t}_{44}\underset{2}{\Gamma}{}^{\bar{l}}_{t\bar{l}} - \underset{2}{\Gamma}{}^{\bar{t}}_{44}\underset{2}{\Gamma}{}^{\bar{l}}_{\bar{t}\bar{l}} - \underset{2}{\Gamma}{}^{t}_{44}\underset{2}{\Gamma}{}^{\bar{4}}_{t\bar{4}} - \underset{2}{\Gamma}{}^{\bar{t}}_{44}\underset{2}{\Gamma}{}^{\bar{4}}_{\bar{t}\bar{4}}\\
&\quad\Rightarrow \underset{2}{R}_{44} =\tfrac{1}{3}\partial_l\partial_l \underset{4}{\gamma}_{44} + \tfrac{1}{3}\partial_{\bar{l}}\partial_{\bar{l}} \underset{4}{\gamma}_{44} - \tfrac{1}{6}\partial_l\partial_l \underset{4}{\gamma}_{tt} - \tfrac{1}{6}\partial_{\bar{l}}\partial_{\bar{l}} \underset{4}{\gamma}_{tt}\\
&\qquad\qquad -\tfrac{2}{3}\partial_4\partial_4 \underset{2}{\gamma}_{44} + \tfrac{1}{3}\partial_{\bar{4}}\partial_{\bar{4}} \underset{2}{\gamma}_{44} - \partial_4\partial_l \underset{3}{\gamma}_{4l} - \partial_4\partial_{\bar{l}} \underset{3}{\gamma}_{4\bar{l}} - \partial_4\partial_{\bar{4}} \underset{2}{\gamma}_{4\bar{4}}\\
&\qquad\qquad -\tfrac{2}{9}\partial_l \underset{2}{\gamma}_{44}\partial_l \underset{2}{\gamma}_{44} - \tfrac{2}{9}\partial_{\bar{l}} \underset{2}{\gamma}_{44}\partial_{\bar{l}} \underset{2}{\gamma}_{44} - \tfrac{1}{2}\partial_l \underset{2}{\gamma}_{4\bar{4}}\partial_l \underset{2}{\gamma}_{4\bar{4}} - \tfrac{1}{2}\partial_{\bar{l}} \underset{2}{\gamma}_{4\bar{4}}\partial_{\bar{l}} \underset{2}{\gamma}_{4\bar{4}}
\end{aligned}
$$

$$\begin{aligned}
\underset{4}{R}=&\delta_{lt}\underset{4}{R}_{lt}+\underset{4}{R}_{44}+\delta_{\bar{l}\bar{t}}\underset{4}{R}_{\bar{l}\bar{t}}+\underset{4}{R}_{\bar{4}\bar{4}}\\
\Rightarrow \underset{4}{R}=&-\partial_l\partial_t\underset{4}{\gamma}_{lt}-\partial_{\bar{l}}\partial_{\bar{t}}\underset{4}{\gamma}_{\bar{l}\bar{t}}-2\partial_l\partial_{\bar{t}}\underset{4}{\gamma}_{l\bar{t}}-\tfrac{1}{3}\partial_l\partial_l\underset{4}{\gamma}_{44}-\tfrac{1}{3}\partial_{\bar{l}}\partial_{\bar{l}}\underset{4}{\gamma}_{44}-\tfrac{1}{3}\partial_l\partial_l\underset{4}{\gamma}_{tt}-\tfrac{1}{3}\partial_{\bar{l}}\partial_{\bar{l}}\underset{4}{\gamma}_{tt}\\
&-\tfrac{4}{3}\partial_4\partial_4\underset{2}{\gamma}_{44}-\tfrac{4}{3}\partial_{\bar{4}}\partial_{\bar{4}}\underset{2}{\gamma}_{44}-\tfrac{10}{9}\partial_l\underset{2}{\gamma}_{44}\partial_l\underset{2}{\gamma}_{44}-\tfrac{10}{9}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\\
&-\tfrac{3}{2}\partial_l\underset{2}{\gamma}_{4\bar{4}}\partial_l\underset{2}{\gamma}_{4\bar{4}}-\tfrac{3}{2}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}-\underset{2}{\gamma}_{4\bar{4}}\partial_l\partial_l\underset{2}{\gamma}_{4\bar{4}}-\underset{2}{\gamma}_{4\bar{4}}\partial_{\bar{l}}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\\
&-2\partial_l\partial_4\underset{3}{\gamma}_{l4}-2\partial_{\bar{l}}\partial_{\bar{4}}\underset{3}{\gamma}_{\bar{l}\bar{4}}-2\partial_l\partial_{\bar{4}}\underset{3}{\gamma}_{l\bar{4}}-2\partial_{\bar{l}}\partial_{\bar{4}}\underset{3}{\gamma}_{\bar{l}\bar{4}}-2\partial_4\partial_{\bar{4}}\underset{2}{\gamma}_{4\bar{4}}
\end{aligned}$$

For the components of the fields equations, the following expressions are needed:

$$\begin{aligned}
\underset{4}{G}_{ab}=&\underset{4}{R}_{ab}-\tfrac{1}{2}\delta_{ab}\underset{4}{R}\\
\Rightarrow \underset{4}{G}_{ab}=&\tfrac{1}{2}\partial_l\left(\partial_l\underset{4}{\gamma}_{ab}-\partial_b\underset{4}{\gamma}_{al}\right)+\tfrac{1}{2}\partial_l\left(\delta_{ab}\partial_t\underset{4}{\gamma}_{lt}-\delta_{al}\partial_t\underset{4}{\gamma}_{bt}\right)+\tfrac{1}{2}\partial_{\bar{l}}\left(\partial_{\bar{l}}\underset{4}{\gamma}_{ab}-\partial_b\underset{4}{\gamma}_{a\bar{l}}\right)+\tfrac{1}{2}\partial_l\left(\delta_{ab}\partial_{\bar{t}}\underset{4}{\gamma}_{l\bar{t}}-\delta_{al}\partial_{\bar{t}}\underset{4}{\gamma}_{b\bar{t}}\right)\\
&+\tfrac{1}{2}\partial_l\left(\delta_{ab}\partial_4\underset{3}{\gamma}_{l4}-\delta_{al}\partial_4\underset{3}{\gamma}_{b4}\right)+\tfrac{1}{2}\partial_l\left(\delta_{ab}\partial_{\bar{4}}\underset{3}{\gamma}_{l\bar{4}}-\delta_{al}\partial_{\bar{4}}\underset{3}{\gamma}_{b\bar{4}}\right)+\tfrac{1}{2}\delta_{ab}\partial_{\bar{l}}\left(\partial_t\underset{4}{\gamma}_{t\bar{l}}+\partial_4\underset{3}{\gamma}_{4\bar{l}}+\partial_{\bar{t}}\underset{4}{\gamma}_{\bar{t}\bar{l}}+\partial_{\bar{4}}\underset{3}{\gamma}_{\bar{4}\bar{l}}\right)\\
&-\tfrac{1}{2}\partial_b\partial_4\underset{3}{\gamma}_{a4}-\tfrac{1}{3}\partial_a\underset{2}{\gamma}_{44}\partial_b\underset{2}{\gamma}_{44}-\tfrac{2}{3}\underset{2}{\gamma}_{44}\partial_a\partial_b\underset{2}{\gamma}_{44}+\tfrac{1}{2}\delta_{ab}\partial_l\underset{2}{\gamma}_{44}\partial_l\underset{2}{\gamma}_{44}+\tfrac{1}{2}\delta_{ab}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\\
&-\tfrac{1}{2}\partial_b\partial_{\bar{4}}\underset{3}{\gamma}_{a\bar{4}}-\tfrac{1}{2}\partial_a\underset{2}{\gamma}_{4\bar{4}}\partial_b\underset{2}{\gamma}_{4\bar{4}}-\underset{2}{\gamma}_{4\bar{4}}\partial_a\partial_b\underset{2}{\gamma}_{4\bar{4}}+\tfrac{3}{4}\delta_{ab}\partial_l\underset{2}{\gamma}_{4\bar{4}}\partial_l\underset{2}{\gamma}_{4\bar{4}}+\tfrac{3}{4}\delta_{ab}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\\
&+\tfrac{1}{2}\delta_{ab}\partial_4\left(\partial_l\underset{3}{\gamma}_{l4}+\partial_4\underset{2}{\gamma}_{44}+\partial_{\bar{l}}\underset{3}{\gamma}_{\bar{l}4}+\partial_{\bar{4}}\underset{2}{\gamma}_{\bar{4}4}\right)+\tfrac{1}{2}\delta_{ab}\partial_{\bar{4}}\left(\partial_l\underset{3}{\gamma}_{l\bar{4}}+\partial_4\underset{2}{\gamma}_{4\bar{4}}+\partial_{\bar{l}}\underset{3}{\gamma}_{\bar{l}\bar{4}}+\partial_{\bar{4}}\underset{2}{\gamma}_{\bar{4}\bar{4}}\right)
\end{aligned}$$

If we impose the conditions (8) and (11), we obtain:

$$\begin{aligned}
\underset{4}{G}_{ab}=&\tfrac{1}{2}\partial_l\left(\partial_l\underset{4}{\gamma}_{ab}-\partial_b\underset{4}{\gamma}_{al}\right)+\tfrac{1}{2}\partial_l\left(\delta_{ab}\partial_t\underset{4}{\gamma}_{lt}-\delta_{al}\partial_t\underset{4}{\gamma}_{bt}\right)+\tfrac{1}{2}\partial_{\bar{l}}\left(\partial_{\bar{l}}\underset{4}{\gamma}_{ab}-\partial_b\underset{4}{\gamma}_{a\bar{l}}\right)+\tfrac{1}{2}\partial_l\left(\delta_{ab}\partial_{\bar{t}}\underset{4}{\gamma}_{l\bar{t}}-\delta_{al}\partial_{\bar{t}}\underset{4}{\gamma}_{b\bar{t}}\right)\\
&+\tfrac{1}{2}\partial_l\left(\delta_{ab}\partial_4\underset{3}{\gamma}_{l4}-\delta_{al}\partial_4\underset{3}{\gamma}_{b4}\right)\\
&-\tfrac{1}{2}\partial_b\partial_4\underset{3}{\gamma}_{a4}-\tfrac{1}{3}\partial_a\underset{2}{\gamma}_{44}\partial_b\underset{2}{\gamma}_{44}-\tfrac{2}{3}\underset{2}{\gamma}_{44}\partial_a\partial_b\underset{2}{\gamma}_{44}+\tfrac{1}{2}\delta_{ab}\partial_l\underset{2}{\gamma}_{44}\partial_l\underset{2}{\gamma}_{44}+\tfrac{1}{2}\delta_{ab}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\\
&\qquad-\tfrac{1}{2}\partial_a\underset{2}{\gamma}_{4\bar{4}}\partial_b\underset{2}{\gamma}_{4\bar{4}}-\underset{2}{\gamma}_{4\bar{4}}\partial_a\partial_b\underset{2}{\gamma}_{4\bar{4}}+\tfrac{3}{4}\delta_{ab}\partial_l\underset{2}{\gamma}_{4\bar{4}}\partial_l\underset{2}{\gamma}_{4\bar{4}}+\tfrac{3}{4}\delta_{ab}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}
\end{aligned}$$

From now on, the argument proceeds as in the case of the mass and charge conservation laws. If we form the integrals of $\underset{4}{G}_{ab}\eta_b$ over a closed surface S, then these integrals must vanish. We have:

$$\left.\begin{aligned}
&\oint_S\underset{4}{G}_{ab}\eta_b dS=0\Rightarrow\\
&\tfrac{1}{2}\overbrace{\oint_S\partial_l\left(\partial_l\underset{4}{\gamma}_{ab}-\partial_b\underset{4}{\gamma}_{al}\right)\eta_b dS}^{=0}+\tfrac{1}{2}\overbrace{\oint_S\partial_l\left(\delta_{ab}\partial_t\underset{4}{\gamma}_{lt}-\delta_{al}\partial_t\underset{4}{\gamma}_{bt}\right)\eta_b dS}^{=0}+\tfrac{1}{2}\overbrace{\oint_S\partial_{\bar{l}}\left(\partial_{\bar{l}}\underset{4}{\gamma}_{ab}-\partial_b\underset{4}{\gamma}_{a\bar{l}}\right)\eta_b dS}^{=0}\\
&+\tfrac{1}{2}\overbrace{\partial_l\left(\delta_{ab}\partial_{\bar{t}}\underset{4}{\gamma}_{l\bar{t}}-\delta_{al}\partial_{\bar{t}}\underset{4}{\gamma}_{b\bar{t}}\right)\eta_b dS}^{=0}+\tfrac{1}{2}\overbrace{\oint_S\partial_l\left(\delta_{ab}\partial_4\underset{3}{\gamma}_{l4}-\delta_{al}\partial_4\underset{3}{\gamma}_{b4}\right)\eta_b dS}^{=0}\\
&+\oint_S\Big(-\tfrac{1}{2}\partial_b\partial_4\underset{3}{\gamma}_{a4}-\tfrac{1}{3}\partial_a\underset{2}{\gamma}_{44}\partial_b\underset{2}{\gamma}_{44}-\tfrac{2}{3}\underset{2}{\gamma}_{44}\partial_a\partial_b\underset{2}{\gamma}_{44}+\tfrac{1}{2}\delta_{ab}\partial_l\underset{2}{\gamma}_{44}\partial_l\underset{2}{\gamma}_{44}+\tfrac{1}{2}\delta_{ab}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\partial_{\bar{l}}\underset{2}{\gamma}_{44}\\
&\qquad-\tfrac{1}{2}\partial_a\underset{2}{\gamma}_{4\bar{4}}\partial_b\underset{2}{\gamma}_{4\bar{4}}-\underset{2}{\gamma}_{4\bar{4}}\partial_a\partial_b\underset{2}{\gamma}_{4\bar{4}}+\tfrac{3}{4}\delta_{ab}\partial_l\underset{2}{\gamma}_{4\bar{4}}\partial_l\underset{2}{\gamma}_{4\bar{4}}+\tfrac{3}{4}\delta_{ab}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\partial_{\bar{l}}\underset{2}{\gamma}_{4\bar{4}}\Big)\eta_b dS=0
\end{aligned}\right| \qquad (13)$$

We shall choose for S a small spherical surface with the radius R, the center of which is the *pth* charged mass point. We have:

$$\begin{aligned}
&\eta_a = \frac{1}{R}\left(z^a - \overset{p}{y}^a\right) &&, \quad \eta_l\eta_l = 1\\
&\partial_a R = \eta_a &&, \quad \partial_4 R = -\partial_4 \overset{p}{y}^l\ \eta_l\\
&\partial_b \eta_a = \frac{1}{R}\left(\delta_{ab} - \eta_a\eta_b\right) &&, \quad \partial_4 \eta_a = \frac{1}{R}\left(\eta_a\eta_l - \delta_{al}\right)\partial_4 \overset{p}{y}^l
\end{aligned}$$

We may rewrite the solutions (12):

$$\begin{aligned}
\underset{2}{\gamma_{44}} &= -\overset{p}{\mu}\left(\frac{1}{R}+\frac{1}{\bar{R}}\right) - \sum_{k=1}^{N,k\neq p}\overset{k}{\mu}\left(\frac{1}{\overset{k}{r}}+\frac{1}{\overset{k}{\bar{r}}}\right)\\
\underset{3}{\gamma_{a4}} &= -\overset{p}{\mu}\left(\frac{\partial_4\overset{p}{y}^a}{R}+\frac{\partial_{\bar{4}}\overset{p}{\bar{y}}^a}{\bar{R}}\right) - \sum_{k=1}^{N,k\neq p}\overset{k}{\mu}\left(\frac{\partial_4\overset{k}{y}^a}{\overset{k}{r}}+\frac{\partial_{\bar{4}}\overset{k}{\bar{y}}^a}{\overset{k}{\bar{r}}}\right)\\
\underset{2}{\gamma_{4\bar{4}}} &= +\overset{p}{\rho}\left(\frac{1}{R}+\frac{1}{\bar{R}}\right) + \sum_{k=1}^{N,k\neq p}\overset{k}{\rho}\left(\frac{1}{\overset{k}{r}}+\frac{1}{\overset{k}{\bar{r}}}\right)\\
\underset{3}{\gamma_{a\bar{4}}} &= +\overset{p}{\rho}\left(\frac{\partial_4\overset{p}{y}^a}{R}+\frac{\partial_{\bar{4}}\overset{p}{\bar{y}}^a}{\bar{R}}\right) + \sum_{k=1}^{N,k\neq p}\overset{k}{\rho}\left(\frac{\partial_4\overset{k}{y}^a}{\overset{k}{r}}+\frac{\partial_{\bar{4}}\overset{k}{\bar{y}}^a}{\overset{k}{\bar{r}}}\right)
\end{aligned}$$

We have:

$$\overset{k}{r} = \sqrt{\left(z^l - \overset{k}{y}^l\right)\left(z^l - \overset{k}{y}^l\right)} = \overset{k}{r} = \sqrt{\left(\overset{p,k}{y}^l + R\eta_l\right)\left(\overset{p,k}{y}^l + R\eta_l\right)} \text{ with } \overset{p,k}{y}^l = \overset{p}{y}^l - \overset{k}{y}^l$$

$$\Rightarrow \overset{k}{r} = \overset{p,k}{r}\sqrt{\left(1+\frac{2\,\overset{p,k}{y}^l\ \eta_l R}{(\overset{p,k}{r})^2}+\frac{R^2}{(\overset{p,k}{r})^2}\right)} \text{ with } \overset{p,k}{r} = \sqrt{\left(\overset{p,k}{y}^l\right)\left(\overset{p,k}{y}^l\right)} \Rightarrow \overset{k}{r} = \overset{p,k}{r}\left(1+\frac{\overset{p,k}{y}^l\ \eta_l}{(\overset{p,k}{r})^2}R+\cdots\right)$$

We therefore obtain:

$$\begin{aligned}
\underset{2}{\gamma_{44}} &= -\overset{p}{\mu}\left(\frac{1}{R}+\frac{1}{\bar{R}}\right) - \sum_{k=1}^{N,k\neq p}\overset{k}{\mu}\left[\frac{1}{\overset{p,k}{r}}\left(1-\frac{\overset{p,k}{y}^l\ \eta_l}{(\overset{p,k}{r})^2}R+\cdots\right)+\frac{1}{\overset{p,k}{\bar{r}}}\left(1-\frac{\overset{p,k}{\bar{y}}^l\ \bar{\eta}_l}{(\overset{p,k}{\bar{r}})^2}\bar{R}+\cdots\right)\right]\\
\underset{3}{\gamma_{a4}} &= -\overset{p}{\mu}\left(\frac{\partial_4\overset{p}{y}^a}{R}+\frac{\partial_{\bar{4}}\overset{p}{\bar{y}}^a}{\bar{R}}\right) - \sum_{k=1}^{N,k\neq p}\overset{k}{\mu}\left[\frac{\partial_4\overset{k}{y}^a}{\overset{p,k}{r}}\left(1-\frac{\overset{p,k}{y}^l\ \eta_l}{(\overset{p,k}{r})^2}R+\cdots\right)+\frac{\partial_{\bar{4}}\overset{k}{\bar{y}}^a}{\overset{p,k}{\bar{r}}}\left(1-\frac{\overset{p,k}{\bar{y}}^l\ \bar{\eta}_l}{(\overset{p,k}{\bar{r}})^2}\bar{R}+\cdots\right)\right]\\
\underset{2}{\gamma_{4\bar{4}}} &= +\overset{p}{\rho}\left(\frac{1}{R}+\frac{1}{\bar{R}}\right) + \sum_{k=1}^{N,k\neq p}\overset{k}{\rho}\left[\frac{1}{\overset{p,k}{r}}\left(1-\frac{\overset{p,k}{y}^l\ \eta_l}{(\overset{p,k}{r})^2}R+\cdots\right)+\frac{1}{\overset{p,k}{\bar{r}}}\left(1-\frac{\overset{p,k}{\bar{y}}^l\ \bar{\eta}_l}{(\overset{p,k}{\bar{r}})^2}\bar{R}+\cdots\right)\right]\\
\underset{3}{\gamma_{a\bar{4}}} &= +\overset{p}{\rho}\left(\frac{\partial_4\overset{p}{y}^a}{R}+\frac{\partial_{\bar{4}}\overset{p}{\bar{y}}^a}{\bar{R}}\right) + \sum_{k=1}^{N,k\neq p}\overset{k}{\rho}\left[\frac{\partial_4\overset{k}{y}^a}{\overset{p,k}{r}}\left(1-\frac{\overset{p,k}{y}^l\ \eta_l}{(\overset{p,k}{r})^2}R+\cdots\right)+\frac{\partial_{\bar{4}}\overset{k}{\bar{y}}^a}{\overset{p,k}{\bar{r}}}\left(1-\frac{\overset{p,k}{\bar{y}}^l\ \bar{\eta}_l}{(\overset{p,k}{\bar{r}})^2}\bar{R}+\cdots\right)\right]
\end{aligned}$$

For the values of the integrals (13) do not depend on the size of S, they must be independent of R. If the integrands are expanded into power series with respect to R, then all those terms which contain R in any power other than R^{-2} cannot contribute to the integrals. We shall therefore reduce the necessary computations by expanding all terms in the integrands into power series and by neglecting all expressions which are not multiplied by R^{-2} . We have:

$$\partial_b \partial_4 \gamma_{\underset{3}{a4}} = +\frac{\overset{p}{\mu}}{R^2} \eta_b \partial_4 \partial_4 \overset{p}{y}{}^a + \text{other terms which are not multiplied by } R^{-2} .$$

$$\partial_a \gamma_{\underset{2}{44}} \partial_b \gamma_{\underset{2}{44}} = + \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \left(\eta_a \overset{p,k}{y}{}^b + \eta_b \overset{p,k}{y}{}^a \right) + \text{other terms which are not multiplied by } R^{-2} .$$

$$\gamma_{\underset{2}{44}} \partial_a \partial_b \gamma_{\underset{2}{44}} = + \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l \left(\delta_{ab} - 3\eta_a \eta_b \right) + \text{other terms which are not multiplied by } R^{-2}$$

.

$$\delta_{ab} \partial_l \gamma_{\underset{2}{44}} \partial_l \gamma_{\underset{2}{44}} = +2\delta_{ab} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l + \text{other terms which are not multiplied by } R^{-2} .$$

$$\delta_{ab} \partial_{\bar{l}} \gamma_{\underset{2}{44}} \partial_{\bar{l}} \gamma_{\underset{2}{44}} = 0 + \text{other terms which are not multiplied by } R^{-2} .$$

$$\partial_a \gamma_{\underset{2}{4\bar{4}}} \partial_b \gamma_{\underset{2}{4\bar{4}}} = - \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \left(\eta_a \overset{p,k}{y}{}^b + \eta_b \overset{p,k}{y}{}^a \right) + \text{other terms which are not multiplied by } R^{-2} .$$

$$\gamma_{\underset{2}{4\bar{4}}} \partial_a \partial_b \gamma_{\underset{2}{4\bar{4}}} = - \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l \left(\delta_{ab} - 3\eta_a \eta_b \right) + \text{other terms which are not multiplied by } R^{-2}$$

.

$$\delta_{ab} \partial_l \gamma_{\underset{2}{4\bar{4}}} \partial_l \gamma_{\underset{2}{4\bar{4}}} = -2\delta_{ab} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l + \text{other terms which are not multiplied by } R^{-2} .$$

$$\delta_{ab} \partial_{\bar{l}} \gamma_{\underset{2}{4\bar{4}}} \partial_{\bar{l}} \gamma_{\underset{2}{4\bar{4}}} = 0 + \text{other terms which are not multiplied by } R^{-2} .$$

We therefore obtain:

$$\begin{aligned}
\underset{4}{G}_{ab} = & -\tfrac{1}{2} \frac{\overset{p}{\mu}}{R^2} \eta_b \partial_4 \partial_4 \overset{p}{y}{}^a \\
& -\tfrac{1}{3} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \left(\eta_a \overset{p,k}{y}{}^b + \eta_b \overset{p,k}{y}{}^a \right) - \tfrac{2}{3} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l \left(\delta_{ab} - 3\eta_a \eta_b \right) + \delta_{ab} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l \\
& +\tfrac{1}{2} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \left(\eta_a \overset{p,k}{y}{}^b + \eta_b \overset{p,k}{y}{}^a \right) + \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l \left(\delta_{ab} - 3\eta_a \eta_b \right) - \tfrac{3}{2} \delta_{ab} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \eta_l \overset{p,k}{y}{}^l \\
& + \text{other terms which do not contribute to the integrals (13)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \underset{4}{G}_{ab} \eta_b = & -\tfrac{1}{2} \frac{\overset{p}{\mu}}{R^2} \partial_4 \partial_4 \overset{p}{y}{}^a + \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \left(-\tfrac{1}{3} \overset{p,k}{y}{}^a + 2\eta_a \eta_l \overset{p,k}{y}{}^l \right) + \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \frac{1}{R^2} \left(+\tfrac{1}{2} \overset{p,k}{y}{}^a - 3\eta_a \eta_l \overset{p,k}{y}{}^l \right) \\
& + \text{other terms which do not contribute to the integrals (13)}
\end{aligned}$$

We have:

$$\oint_S \eta_l \overset{p,k}{y^l} \eta_a dS = \tfrac{4}{3}\pi R^2 \overset{p,k}{y^a} \quad \text{and} \quad \oint_S dS = 4\pi R^2$$

We therefore obtain:

$$\oint_S \underset{4}{G_{ab}}\eta_b dS = -2\pi \overset{p}{\mu}\, \partial_4\partial_4 \overset{p}{y^a} + \tfrac{4}{3}\pi \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \overset{p,k}{y^a} - 2\pi \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \overset{p,k}{y^a} = 0$$

$$\Rightarrow - \overset{p}{\mu}\, \partial_4\partial_4 \overset{p}{y^a} + \tfrac{2}{3} \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\mu}\overset{k}{\mu}}{(\overset{p,k}{r})^3} \overset{p,k}{y^a} - \sum_{k=1}^{N,k\neq p} \frac{\overset{p}{\rho}\overset{k}{\rho}}{(\overset{p,k}{r})^3} \overset{p,k}{y^a} = 0$$

If we pose:

$$\overset{k}{\mu} = \tfrac{3}{2}G \overset{k}{m} \quad \text{and} \quad \overset{k}{\rho} = \sqrt{\tfrac{3}{2}GK}\, \overset{k}{q} \tag{14}$$

We obtain, for $dz^4 = icdt$, the classical equations of motion for charged mass points:

$$\overset{p}{m}\overset{p}{\ddot{y}^a} + \sum_{k=1}^{N,k\neq p} G \frac{\overset{p}{m}\overset{k}{m}}{(\overset{p,k}{r})^3} \overset{p,k}{y^a} - \sum_{k=1}^{N,k\neq p} K \frac{\overset{p}{q}\overset{k}{q}}{(\overset{p,k}{r})^3} \overset{p,k}{y^a} = 0 \tag{15}$$

We recognize in (15) the Newton inertial force, the Newton gravitational force and the Coulomb electrostatic force. We conclude from (12) and (15) that $\underset{2}{\gamma_{44}}$ represents the gravitational potential and that $\underset{2}{\gamma_{4\bar{4}}}$ represents the electrostatic potential. Therefore, we conclude from the equation $\partial_l \underset{3}{\gamma_{l\bar{4}}} + \partial_4 \underset{2}{\gamma_{4\bar{4}}} = 0$ of (10) that $\underset{2}{\gamma_{\alpha\bar{4}}}$ represent the electromagnetic potentials A_a. Finally, we conclude from (14) that the electromagnetic field could not exist without the gravitational field ($\overset{k}{\rho} = 0$ if $G = 0$).

12 conclusion

We have shown that the linearization of the equations (6) leads to the "Newton-Maxwell-Lorentz" results of classical physics. That is, the gravitational field and the electromagnetic field of charged mass points and their equations of motion. We have also seen that the formal conditions (11) must be imposed to the solutions of the linearized equations (10) to achieve these results.

The formal conditions (11) cannot be directly imposed to the solutions of the equations (6). This mean that in the most general case, a physical interpretation must be given to the double spacetime being considered. We believe that this can only be achieved by finding exact solutions (generalized Schwarzschild solutions) to the equations (6). We did not try to achieve this result in this essay.

13 References

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